Competitive Nonlinear Pricing and Contract Variety*

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Abstract

We analyze markets with both horizontally and vertically differentiated products under both monopoly and duopoly. In the base model with two consumer types, we identify conditions under which entry prompts an incumbent to expand or contract its low end of the product line. Our analysis offers a novel explanation for the widespread use of "fighting brands" and "product line pruning". We also extend our analysis to asymmetric firms and three types of consumers, and show that depending on specific environment, entry may lead the incumbent to expand or contract its middle range of the product line (middle contracts). Our results are mainly driven by interactions between horizontal differentiation (competition) and vertical screening of consumers.

Keywords: Contract, contract variety, product line, nonlinear pricing, fighting brands, product line pruning.

1 Introduction

In response to entry or increased competition, incumbents often expand their product lines to introduce new products, or contract their product lines to remove some existing products. When the expansion and contraction occur on the low end of the market or product line, they are known as

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"*fighting brands*" and "*product line pruning*", respectively (Johnson and Myatt, 2003). Both fighting brands and product line pruning are common in business. A notable successful fighting brand is Intel's Celeron processor. Despite the success of its Pentium chips, Intel faced a major threat from AMD's K6 chips that were cheaper and better placed to serve the emerging low-cost PC market. Intel wanted to protect the brand equity and price premium of its Pentium chips, but also wanted to prevent AMD from obtaining a foothold into the lower end of the market. Intel thus introduced Celeron as a cheaper, less powerful version of its Pentium chips to serve this market. Intel's 80% share of the global PC market is testament to the potential of a successful fighting brand to restrict competitors and open up additional segments of the market. Other cases of fighting brands abound. For example, Qantas (Australia) launched JetStar to take on Virgin Blue, British Airways (UK) launched GO to take on Ryanair and EasyJet, GM launched Saturn to take on Japanese imports into America, Merck (Germany) launched Zocor MSD to take on generic brands and protect Zocor in Europe, and Philip Morris (Russia) launched Bond Street to take on local brands and protect Marlboro cigarette. Product line pruning is also widespread. For example, in response to private label brands in the early 1990's, Procter & Gamble removed some weak products from its product line. Facing the entry of Titan into the low end of the market, Timex removed a number of its lower-priced watches from the Indian market.

Note that incumbent firms do not only respond to competition by adjusting the low end of their product lines; they may adjust the middle range of product lines as well. On the one hand, increased competition may lead to addition of middle products. For example, following the release of TomTom's first GPS series GO in March 2004, the incumbent Garmin introduced the Quest series as a medium-level product which featured a 2.7" 240x160 non-touch-sensitive color screen. On the other hand, increased competition may lead to the removal of some existing middle products. For example, after OpenOffice's entry to release OpenOffice.org 1.0 for free on May 1, 2002, on November 17, 2003 Office 2003 removed three variations of the Professional edition, i.e., Professional with FrontPage, Professional with Publisher, and Professional Special, leaving only the Professional edition that contains all the packages.

The examples above show that incumbent firms may respond to competition by adjusting their product lines either at the low end or in the middle range. In this research, we offer a framework to analyze how increased competition affects the product line or the variety of contracts offered.

Specifically, in our model consumers are both vertically and horizontally differentiated: in the vertical dimension they have different marginal utilities about quality and in the horizontal dimension they have different tastes for firms' products. Firms' products are horizontally differentiated, and in the vertical dimension each firm offers a range of products with different qualities. Under duopoly firms compete by offering a menu of contracts (or, equivalently, price-quality schedules). The driving force in our model is the interaction between horizontal differentiation (competition) and the vertical screening (on vertical types).

In the base model, we focus on the case where consumers only have two vertical types, h (high) and l (low). We compare the optimal menu of contracts under monopoly to the equilibrium menu of contracts in the symmetric equilibrium under duopoly. Our main result is that when the degree of horizontal differentiation (measured by the per unit transportation cost k) is low, entry will never lead to product line pruning, but it may lead to fighting brands; on the other hand, when the degree of horizontal differentiation is high, entry will never lead to fighting brands, but it may lead to product line pruning. Similar results hold when further entry occurs.[1] If the initial degree of competition is strong, further entry can only lead to fighting brands; if the initial degree of competition is weak, further entry can only lead to product line pruning.

The intuition of the above results is as follows. When the degree of horizontal differentiation (k) is low ($k \leqslant 1/2$) or the initial competition level is strong (in the n-firm case), under monopoly, in order to reduce the informational rent enjoyed by type h the low type may be excluded, in which case only an h contract is offered. Under duopoly, competition for type h leads to higher rent for type h. This relaxes the incentive compatibility constraint (or the screening condition) along the vertical dimension. Hence the information rent consideration becomes less important as type h will secure higher rent anyway. This implies that offering a contract to low-type consumers has a smaller negative impact on the profitability from the high type. As a result, offering a contract to low-type consumers can be profitable, leading to fighting brands. The reason that entry may lead to product line pruning, however, has to do with the effect on market share, in addition to the above competition effect. Entry reduces the incumbent firm's market share for type h, which tends to reduce the rent provision to type h. When k is relatively large (or initial competition level is weak in n-firm case),

[1] In Section 2.4, we demonstrate that our analysis of two-type case can be easily translated to the case with n firms, which is standard for a Salop circular city model.

under duopoly neither firm has an incentive to steal the other firm's market share for type h, thus entry leads to a lower rent provision to type h. This makes the screening condition more binding in the vertical dimension. In the mean time, there is a mitigating effect that under duopoly the measure of h-type consumers served by each firm is less than that served under monopoly, which makes firms more willing to increase the rent to low-type consumers. We can identify conditions under which the first effect dominates (hence firms are more concerned about the information rent consideration under duopoly), leading to product line pruning for the incumbent to remove its low-quality product.

We then extend our analysis to asymmetric firms, where the entrant is technologically inferior to the incumbent in the sense that the upper bound of the quality range for the entrant's products is lower than that of the incumbent. We first show that if both firms offer two contracts, the quality distortion of the low contract offered by the incumbent is smaller. We also provide comparative statics with respect to the degree of technology asymmetry, and show that when k is sufficiently small (so that there is effective competition for the high type), as the entrant becomes more inferior, fighting brands become less likely and product pruning becomes more likely.

Finally, we extend our analysis to the case with symmetric firms and three vertical types, h, m, and l. Unlike the two-type model, (partial) pooling may now occur: the middle and the low types choose the same contract. When the degree of horizontal differentiation is low, we identify conditions under which a monopolist offers two contracts, one targeting the high type, and the other targeting both the low and the middle types (pooling), while in duopoly both firms offer three contracts, targeting each type separately (fully separating). In other words, entry leads to the addition of the middle contract (the middle quality product). When the degree of horizontal differentiation is high, on the other hand, we identify conditions under which the opposite occurs: a monopoly firm offers fully separating contracts, but in duopoly, each firm offers only two contracts, with the middle and low types pooled at the low contract. In this case, entry leads to the removal of a middle contract or a middle quality product.

While it is fairly common for incumbent firms to respond to competition by adjusting product lines, the connection between competition and product line or contract variety has received little attention from economists. Johnson and Myatt (2003, JM hereafter) were the first to develop a formal model and offer an explanation for fighting brands and product line pruning. In their analysis, a single firm enters a market originally dominated by a monopolist. The duopolists then compete in quantities, each potentially offering a range of quality-differentiated products. They show that whether the incumbent will choose to extend or contract its product line depends on the shape of the marginal revenue curves in the market. When marginal revenue is decreasing, the incumbent responds to entry by pruning low-quality products. However, when marginal revenue is increasing in some regions, upon entry an incumbent may find it optimal to introduce a lower-quality product (brand fighting).

Our paper offers an alternative explanation for fighting brands and product line pruning. Our model differs from JM in the following aspects. First, in JM firms compete in quantities, while in our model products are horizontally differentiated and firms compete in prices. Second, in JM the set of (discrete) qualities that firms are able to produce is exogenously determined (though firms choose endogenously to produce a subset of it). In our model, the qualities that firms offer are endogenously determined. Combining the above two features, in our model firms compete by offering a menu of contracts (or, equivalently, price-quality schedules). Therefore, in a sense our model is more comparable to the literature of nonlinear pricing and price discrimination. The modeling difference leads to different implications. In JM, whether fighting brand or product line pruning will occur depends on the shape of the marginal revenue curve, which in turn depends on the distribution of consumer types. In our model, it is the degree of horizontal differentiation (intensity of competition) that determines whether fighting brands or product line pruning will occur. The other key difference is that in JM the changes in product line always happen in the low-end, while in our three-type model we demonstrate that the changes in product line (or contract variety) can occur at the middle range, which accounts for another type of product line adjustment in response to competition.[2]

Since the seminal work of Mussa and Rosen (1978) and Maskin and Riley (1984) on monopolistic nonlinear pricing, there is a growing literature on nonlinear pricing in competitive settings, see, for example, Spulber (1989), Champsaur and Rochet (1989), Wilson (1993), Gilbert and Matutes (1993), Stole (1995), Verboven (1999), Villas-Boas and Schmidt-Mohr (1999), Armstrong and Vickers (2001, 2006), Rochet and Stole (1997, 2002), Ellison (2005). However, all these papers assume that all the (vertical) types of consumers are served in the market. This full market coverage assumption does greatly simplify the analysis, but precludes the effect of competition on the number of contracts offered on vertical dimension, which is central to our analysis.

[2] Johnson and Myatt (2006) extend JM into n-firm setting, which is comparable to our n-firm case.

Technically speaking, our approach is most closely related to Rochet and Stole (2002), who offer a general framework with both horizontally and vertically differentiated products, and discrete and continuos types of consumers (on vertical dimension). This framework covers both monopoly and duopoly cases. They show that in the monopoly case there is either bunching or no quality distortion at the bottom. In the duopoly case, they show that under full market coverage quality distortions disappear and the equilibrium is characterized by the cost-plus-fee pricing feature (a similar result obtained in Armstrong and Vickers, 2001).

Yang and Ye (2008) provide a complementary analysis of Rochet and Stole (2002) to allow for partial market coverage on vertical dimension. By focusing on the case where the lowest type of consumer being served is endogenously determined, they are able to study the effect of varying horizontal differentiation (competition) on the market coverage. However, Yang and Ye (2008) assume a continuous type space along the vertical dimension. This prevents an analysis of how increased competition can affect the number of contracts or product lines offered,[3] a main task left for the current research with discrete (vertical) types of consumers.

The paper is organized as follows. The next section lays down the base model with two types. Section 3 analyzes the case with asymmetric firms, in which the entrant may be technologically inferior to the incumbent. Section 4 extends the base model to the case with three consumer types. Section 5 concludes.

2 The Base Model

We consider a market with both horizontally and vertically differentiated products where consumers' preferences differ in two dimensions. In the horizontal dimension, consumers have different tastes for different products (firms); in the vertical dimension consumers have different marginal utilities over quality. More specifically, in the vertical dimension a consumer is either type h (High) or type l (Low), i.e., the vertical type $\theta \in \{\theta_h, \theta_l\}$, where $\theta_h > \theta_l > 0$. Without loss of generality, we normalize $\theta_h = 1$. The proportions of types h and l are α and $1 - \alpha$, respectively. We model the taste dimension as the horizontal "location" of a consumer on a unit-length circle representing the ideal product for that consumer: we adopt the Salop's circular city model so that in the horizontal

[3] Technically speaking, with a continuum of types, the number of contracts offered is uncountably many.

dimension, each type of consumer is uniformly distributed on a unit-length circumference. The total measure of consumers is 1.

We consider cases with one or two horizontal products. A horizontal product may offer two goods of different qualities q. In the horizontal dimension, each consumer is characterized by d, the distance between his own (ideal) location and the location of a particular product (say, product 1). To sum up, each consumer is characterized by a two-dimensional type (θ, d). Each consumer has a unit demand for the good. If a type θ consumer consumes a product of quality q which is located away from his own location by distance d and pays a transfer t, his utility is given by

$$u(\theta, q, t, d) = \theta q - t - kd, \tag{1}$$

where k measures the degree of horizontal differentiation: it indicates consumers' willingness to buy a good that is not exactly of his own taste, and is the per-unit transportation cost in the standard Hotelling or Salop's circular city models. We assume that there is a minimum quality requirement so that each firm can only produce $q \geq \underline{q}$, $\underline{q} \in (0, \theta_l)$. This requirement is due to either government regulation or the firm's technology constraint. It may also arise from the possibility that any product with $q < \underline{q}$ is dysfunctional, which gives zero or even negative utility to consumers. The reason that we introduce the minimum quality requirement will be discussed shortly.

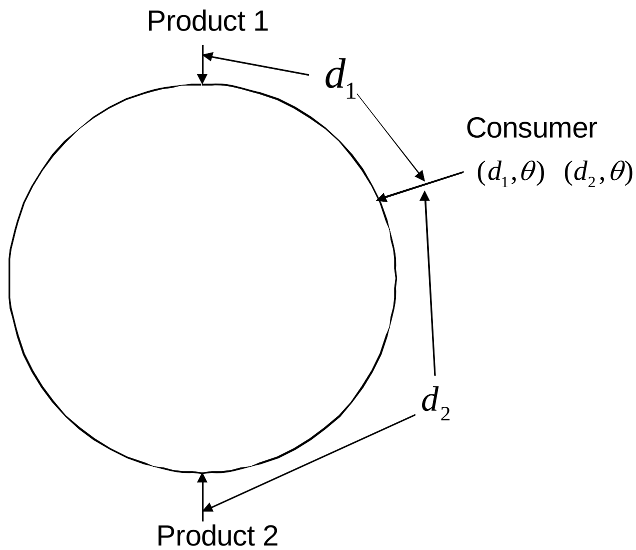
If a firm sells a product of quality q to a consumer, its profit from that sale is given by

$$v(q, t) = t - \frac{1}{2}q^2,$$

where $\frac{1}{2}q^2$ is the cost of producing a good of quality q.[4]

Neither θ nor d is observable to firms, but as is obvious from (1) the single crossing property is only satisfied in the vertical dimension. As a result firms can only make offers to sort consumers with respect to their vertical types in our model. We are interested in how market structure affects the products offered in the vertical dimension. Specifically, we compare two different scenarios. The first scenario is monopoly, where a single horizontal product is offered by a single firm. The second scenario is duopoly, where two horizontal products are offered by two different firms, who evenly split the unit-length circle as illustrated by the following Salop circular model.

[4] The specific form of the cost function should not affect the general insight of our results, though we do require the cost function to be convex.



2.1 Monopoly

The monopolist can possibly offer two qualities (contracts) (q_h, t_h) and (q_l, t_l) targeting types h and l, respectively. Associated with two contracts, the gross utility of a type i, $i = h, l$, who chooses contract (q_i, t_i) is given by $u_i = \theta_i q_i - t_i$. Since it is more convenient to use u_i instead of t_i, we write a contract as (q_i, u_i).[5] A menu of (two) contracts is incentive compatible if and only if:

$$(1-\theta_l)q_h \geq u_h - u_l \geq (1-\theta_l)q_l,$$

where the first inequality is the upward incentive compatibility (UIC) constraint and the second inequality is the downward incentive compatibility (DIC) constraint.

Given u_i, the (half) market share for each type, $M(u_i, i)$,[6] is given by,

$$M(u_h, h) = \alpha \min\left\{\frac{1}{2}, \frac{u_h}{k}\right\}, \text{ and } M(u_l, l) = (1-\alpha)\min\left\{\frac{1}{2}, \frac{u_l}{k}\right\}.$$

The monopolist has two options in terms of contract variety: offering one contract or offering two contracts. Without the minimum quality requirement $q \geq \underline{q}$, it can be readily shown that offering two contracts dominates offering only one contract. With the minimum quality requirement, the above

[5] Here we follow the lead of Armstrong and Vickers (2001), who model firms as supplying utility directly to consumers.

[6] To ease exposition, we use half market share throughout the paper.

property no longer holds. Therefore, we assume a minimum quality requirement in our model, in order to make the choice of contract variety nontrivial, as it is sometimes optimal for the monopolist to only offer one contract. When only one contract is offered, either only type h agents participate or both types participate (pooling). The following lemma establishes that pooling is never optimal.

LEMMA 1 *Suppose the monopolist offers a single contract with $q \in [\underline{q}, 1]$ and both types participate. Then the monopolist can earn higher profit by offering two contracts.*

Proof. See Appendix. ■

Lemma 1 shows that we can focus on offering two contracts or offering h contracts targeting at type h only when searching for optimal contract(s). When two contracts are offered, the UIC is always slack (see Lemma 1 in Rochet and Stole, 2002, for the details). Given that the UIC is slack, the quality provision for type h should be efficient: $q_h = 1$. Moreover, we only need to worry about the DIC. Because of the horizontal differentiation, the DIC might be binding or slack.

We first study the case when a single contract targeting at type h is offered. In that case, the firm's programming problem is:

$$\max_{u_h} \alpha \min\left\{\frac{1}{2}, \frac{u_h}{k}\right\}\left(\frac{1}{2} - u_h\right)$$

It is easily verified that the optimal solution is given by

$$u_h^{fb} = \begin{cases} \frac{1}{4} & \text{if } k \in (\frac{1}{2}, 1] \\ \frac{k}{2} & \text{if } k \in (0, \frac{1}{2}] \end{cases}.$$

The resulting market share for type h is either $1/2$ if $k \leq 1/2$, or $\frac{1}{4k}$ if $k > 1/2$. For competition to be nontrivial in the duopoly case, we assume $k < 1$, so that the market share for type h under monopoly is more than $1/4$.

We next consider the case when two contracts are offered. The firm's programming problem is:

$$\max_{(u_h, q_l, u_l)} \alpha \min\left\{\frac{1}{2}, \frac{u_h}{k}\right\}\left(\frac{1}{2} - u_h\right) + (1-\alpha)\min\left\{\frac{1}{2}, \frac{u_l}{k}\right\}\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right)$$

$$\text{subject to: } u_h \geq u_l + (1-\theta_l)q_l \quad \text{(DIC)}$$

$$q_l \geq \underline{q}, \quad u_l \geq 0$$

The (full-information) unconstrained solution is $u_h = u_h^{fb}$, $q_l = \theta_l$ and

$$u_l^{fb} = \begin{cases} \frac{\theta_l^2}{4} & \text{if } k \in (\frac{\theta_l^2}{2}, 1] \\ \frac{k}{2} & \text{if } k \in (0, \frac{\theta_l^2}{2}] \end{cases}.$$

We now turn our attention to the case in which types are private information. The unconstrained solution is not feasible if $k \leq \frac{\theta_l^2}{2}$. It is not feasible either when

$$\begin{array}{lr} \frac{k}{2} - \frac{1}{4}\theta_l^2 < (1-\theta_l)\theta_l & \text{if } k \in (\frac{\theta_l^2}{2}, \frac{1}{2}] \\ \frac{1}{4}(1-\theta_l^2) < (1-\theta_l)\theta_l \Leftrightarrow \theta_l > \frac{1}{3} & \text{if } k \in (\frac{1}{2}, 1] \end{array}.$$

Combining the above conditions, the unconstrained solution is not feasible if $k < 2\theta_l - \frac{3}{2}\theta_l^2$ in the case $k \leq 1/2$, and $\theta_l > \frac{1}{3}$ in the case $k \in (\frac{1}{2}, 1]$.

The Case that $k \leq \frac{1}{2}$. Note that if $(1-\theta_l)\underline{q} < \frac{k}{2}$, offering two contracts is better than offering h contract alone. To see this, suppose the firm offers the first-best h contract $(1, u_h^{fb})$ alone. Given that $(1-\theta_l)\underline{q} < \frac{k}{2}$, the firm can profitably offer l contract (q_l, u_l) with $q_l \geq \underline{q}$, $u_l \geq 0$, and $u_l + (1-\theta_l)q_l \leq \frac{k}{2}$. Therefore, offering h contract alone cannot be optimal.

Next suppose $(1-\theta_l)\underline{q} \geq \frac{k}{2}$. Note that this implies that the unconstrained solution is not feasible, so the DIC must bind.[7] By DIC, this implies that $u_h \geq k/2$ if two contracts are offered. The programming problem becomes:[8]

$$\max_{(u_h, q_l, u_l)} \frac{\alpha}{2}\left[\frac{1}{2} - u_l - (1-\theta_l)q_l\right] + (1-\alpha)\frac{u_l}{k}\left[\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right]$$
$$\text{subject to: } u_l \geq 0, \quad q_l \geq \underline{q}$$

Let the Lagrangian multiplier for the first and second constraints be μ and λ, respectively. The first order conditions are

$$-\frac{\alpha}{2} + \frac{1-\alpha}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - 2u_l\right) + \mu = 0;\ \mu \geq 0,\ \mu = 0 \text{ if } u_l > 0; \quad (2)$$

$$-\frac{\alpha}{2}(1-\theta_l) + \frac{1-\alpha}{k}u_l(\theta_l - q_l) + \lambda = 0;\ \lambda \geq 0,\ \lambda = 0 \text{ if } q_l > \underline{q}. \quad (3)$$

In general, it is hard to determine the necessary and sufficient conditions under which offering h contract alone is optimal. For this reason we will identify sufficient conditions only. We argue that if $\theta_l^2 < \frac{k\alpha}{1-\alpha}$, offering h contract only is optimal. To see this, note that $\theta_l q_l - \frac{1}{2}q_l^2 \leq \frac{1}{2}\theta_l^2$. Thus $\theta_l^2 < \frac{k\alpha}{1-\alpha}$ implies that the LHS of (2) is negative, which means that $\mu > 0$ and $u_l = 0$. Therefore, the firm has no incentive to offer an l contract.

[7] This is because $(1-\theta_l)\underline{q} < (1-\theta_l)\theta_l < \theta_l - \frac{3}{4}\theta_l^2$.

[8] In writing the following programming problem, we implicitly assumed that $u_l \leq k/2$. Note that in the optimal solution $u_l \leq k/2$, since offering u_l more than $k/2$ will lead to a loss in profit.

Another set of conditions under which offering an h contract alone is optimal is as follows:

$$\alpha(1-\theta_l) > \frac{1-\alpha}{2k}\left(\theta_l^2 - \frac{k\alpha}{1-\alpha}\right)(\theta_l - \underline{q}), \quad (4)$$

$$\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 < \frac{\alpha}{1-\alpha}\frac{k}{2}. \quad (5)$$

To see this, note that condition (4) implies that the LHS of (3) (excluding λ) is negative. Thus $q_l = \underline{q}$ is binding. Now condition (5) implies that the LHS of (2) is negative. Therefore, $u_l = 0$ and the firm has no incentive to offer an l contract.

The case that $k \in (\frac{1}{2}, 1]$. Similar to the previous case, if $(1-\theta_l)\underline{q} < \frac{1}{4}$, then offering two contracts is optimal. The reason is that the firm can always profitably add a low contract without raising u_h when the first-best h contract is offered. Note that $(1-\theta_l)\underline{q} < \frac{1}{4}$ is always satisfied due to the following inequalities:

$$(1-\theta_l)\underline{q} < (1-\theta_l)\theta_l \leq \frac{1}{4}.$$

The first inequality above is due to $\underline{q} < \theta_l$. Therefore, we conclude that when $k \in (\frac{1}{2}, 1]$ two contracts must be offered under monopoly.

The following lemma summarizes the previous analysis.

Lemma 2 *(i) $k \in \left(0, \frac{1}{2}\right)$. If $(1-\theta_l)\underline{q} < \frac{k}{2}$, then the monopolist offers two contracts. If $(1-\theta_l)\underline{q} \geq \frac{k}{2}$, and either $\theta_l^2 < \frac{k\alpha}{1-\alpha}$ or conditions (4) and (5) are satisfied, then offering an h contract only is optimal for the monopolist. (ii) $k \in [\frac{1}{2}, 1]$. Offering two contracts is optimal for the monopolist.*

The result that offering an h contract alone is optimal is due to informational rent considerations. If by offering a low contract too much informational rent needs to be given to high types (relative to the profit from low types), then the firm will optimally exclude low types (by not offering the l contract). From the previous analysis, we see that exclusion is more likely to occur when α is big and $\underline{q}$ is close to θ_l. A bigger α implies that the high type becomes more important. Moreover, when $\underline{q}$ is close to θ_l, the low quality cannot be distorted downward by a large amount, which makes the low-type contract more attractive to the high type. This makes the firm more reluctant to offer a low contract. Part (ii) of Lemma 2 shows that the exclusion of low types is only possible when k is small. When k is big, the firm is willing to give a high rent to type h in order to penetrate enough into the market for the high type. As a result, informational rent consideration becomes less important and the exclusion of the low type becomes less likely.

2.2 Duopoly

Under duopoly, two firms compete by offering contracts (q^j, u^j) with $j = 1, 2$ denoting firm 1 and firm 2, respectively. Firms are symmetric. We adopt Bertrand-Nash equilibrium as our solution concept. Specifically, $\{(q^1, u^1), (q^2, u^2)\}$ is an equilibrium if given (q^{-j}, u^{-j}), firm j maximizes its own profit by choosing (q^j, u^j), $j = 1, 2$. We will focus on symmetric equilibria in which both firms offer the same contract(s), i.e., $q^1 = q^2$ and $u^1 = u^2$.

The result of Lemma 1 can be readily extended to the duopoly setting: there is no equilibrium in which both firms offer one contract and both high and low type consumers are served. In constructing a profitable deviation, we can fix the other firm's contract and let one firm offer another contract targeting either type h or type l, which offers the same utility to the targeting type as the original contract. This means that pooling equilibria do not exist. Therefore, we can concentrate on two possible equilibria. In the first scenario, each firm offers contract h only and only type h consumers are served. In the second scenario, each firm offers two contracts targeting at types h and l separately. Given that $k < 1$, the market for type h will be fully covered in the horizontal dimension. Therefore, the market share for type h of firm 1 becomes $\frac{1}{4} + \frac{u_h^1 - u_h^2}{2k}$. On the other hand, the market for the low type might not be fully covered. As a result, the market share for type l of firm 1 is $\min\left\{\frac{u_l^1}{k}, \frac{1}{4} + \frac{u_l^1 - u_l^2}{2k}\right\}$.

When both firms offer h contracts only, the profit maximization problem for firm 1, given (q_h^2, u_h^2), is as follows:

$$\begin{aligned} \max_{u_h^1} \alpha \left(\frac{1}{4} + \frac{u_h^1 - u_h^2}{2k}\right)\left(\frac{1}{2} - u_h^1\right), \text{ if } u_h^1 + u_h^2 &\geq \frac{k}{2} \\ \max_{u_h^1} \alpha \frac{u_h^1}{k}\left(\frac{1}{2} - u_h^1\right), \text{ if } u_h^1 + u_h^2 &\leq \frac{k}{2} \end{aligned}$$

Note that given u_h^2, firm 1's objective function is not differentiable at $u_h^1 = \frac{k}{2} - u_h^2$. Solving the first case in the above programming problem, we have the equilibrium utility $u_h^D = \frac{1-k}{2}$. However, if $k \in (\frac{2}{3}, 1]$, then $2u_h^D < \frac{k}{2}$. In this case, the second case applies, and the solution is $u_h^D = 1/4$. But then $2u_h^D \geq k/2$ given $k \leq 1$. It turns out that we have a corner solution: $u_h^D = k/4$. To see this, suppose firm 2 offers $u_h^D = k/4$. Then it can be verified that firm 1's profit decreases if u_h^D increases from $k/4$ (the first case), and firm 1's profit decreases as well if u_h^D decreases from $k/4$ (the second case). To sum up, in the symmetric equilibrium, we have

$$u_h^D = \begin{cases} \frac{k}{4} & \text{if } k \in (\frac{2}{3}, 1] \\ \frac{1-k}{2} & \text{if } k \in (0, \frac{2}{3}] \end{cases} .$$

Note that when $k \in (\frac{2}{3}, 1]$, even though the market for type h is fully covered, each firm has no incentive to steal the other firm's market share. In other words, there is no competition between two firms.

Similarly, under full information, $q_l^D = \theta_l$, and u_l^D takes the following form:

$$u_l^D = \begin{cases} \frac{\theta_l^2}{4} & \text{if } k \in [\theta_l^2, 1] \\ \frac{k}{4} & \text{if } k \in [\frac{2}{3}\theta_l^2, \theta_l^2) \\ \frac{\theta_l^2 - k}{2} & \text{if } k \in (0, \frac{2}{3}\theta_l^2) \end{cases} .$$

Competition occurs for type l consumers only when $k \in (0, \frac{2}{3}\theta_l^2)$. When $k \in [\frac{2}{3}\theta_l^2, \theta_l^2)$, although type l consumers are fully covered, there is no competition for type l consumers.

Now suppose that both firms offer contracts h and l. First consider the case $k \in (0, \frac{2}{3}]$. The profit maximization problem for firm 1, given (q_i^2, u_i^2), $i \in \{h, l\}$, is as follows:

$$\max_{(u_h^1, q_l^1, u_l^1)} \alpha \left(\frac{1}{4} + \frac{u_h^1 - u_h^2}{2k}\right)\left(\frac{1}{2} - u_h^1\right) + (1-\alpha) \min\left\{\frac{1}{4} + \frac{u_l^1 - u_l^2}{2k}, \frac{u_l^1}{k}\right\}\left(\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^1\right)$$

$$\text{subject to: } u_h^1 \geq u_l^1 + (1-\theta_l) q_l^1 \quad \text{(DIC)}$$

$$q_l^1 \geq \underline{q};\ u_l \geq 0$$

The Case that $k \in (0, \frac{2}{3}\theta_l^2)$. In this case, from the full-informtion solution we can observe that competition exists for both types, the DIC is slack, and the full-information solution is the equilibrium under private information. Therefore, in duopoly equilibrium firms offer two contracts.

The Case that $(1-\theta_l)\underline{q} < \frac{1-k}{2} = u_h^D$. In this case, in duopoly equilibrium firms must offer two contracts. To see this, suppose in equilibrium each firm just offers an h contract. From the previous analysis the equilibrium utility $u_h^D = \frac{1-k}{2}$. Now given that $(1-\theta_l)\underline{q} < \frac{1-k}{2}$, firm 1 can profitably offer a low contract (q_l, u_l), with $u_l > 0$, $q_l \geq \underline{q}$, and $(1-\theta_l)\underline{q} + u_l \leq u_h^D$ (the DIC is satisfied). Thus offering h contract alone cannot be an equilibrium. Note that if $k \in (0, 1/2]$, $(1-\theta_l)\underline{q} < \frac{1-k}{2}$ since $(1-\theta_l)\underline{q} < \frac{1}{4}$, which is shown earlier. Therefore, when $k \in (0, 1/2]$, in duopoly equilibrium both firms offer two contracts.

The Case that $(1-\theta_l)\underline{q} \geq \frac{1-k}{2}$ and $k \in (\frac{1}{2}, \frac{2}{3}]$. In this case, offering two contracts means that the DIC must bind. Note that this condition implies that $k > \theta_l^2$. To see this, suppose $k \leq \theta_l^2$, then we have

$$(1-\theta_l)\underline{q} < (1-\theta_l)\theta_l < \frac{1-\theta_l^2}{2} \leq \frac{1-k}{2},$$

which contradicts the previous condition. When $k > \theta_l^2$, even under full information the market for type l is not fully covered. Hence under private information the market for type l is not fully covered either. Given (q_i^2, u_i^2), firm 1's programming problem becomes

$$\max_{\{u_h^1, q_l^1, u_l^1\}} \alpha \left[\frac{1}{4} + \frac{u_l^1 + (1-\theta_l)q_l^1 - u_h^2}{2k}\right]\left[\frac{1}{2} - u_l^1 - (1-\theta_l)q_l^1\right] + (1-\alpha)\frac{u_l^1}{k}\left[\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^1\right]$$

$$\text{subject to: } u_l^1 \geq 0, \quad q_l^1 \geq \underline{q}$$

Let the Lagrangian multipliers of the first and second constraints be μ_D and λ_D, respectively. The symmetric equilibrium is characterized by the following first order conditions:

$$\alpha\left[\frac{1}{2k}(\frac{1}{2} - u_h^d) - \frac{1}{4}\right] + \frac{1-\alpha}{k}\left[\theta_l q_l^d - \frac{1}{2}(q_l^d)^2 - 2u_l^d\right] + \mu_D = 0;\ \mu_D \geq 0,\ \mu_D = 0 \text{ if } u_l^d > 0 \quad (6)$$

$$\alpha(1-\theta_l)\left[\frac{1}{2k}(\frac{1}{2} - u_h^d) - \frac{1}{4}\right] + \frac{1-\alpha}{k}u_l^d(\theta_l - q_l^d) + \lambda_D = 0;\ \lambda_D \geq 0,\ \lambda_D = 0 \text{ if } q_l^d > \underline{q} \quad (7)$$

The Case that $k \in (\frac{2}{3}, 1]$. In this case, recall that under full information, $u_h^D = \frac{k}{4}$, and although the market for type h is fully covered there is no competition for type h. If $(1-\theta_l)\underline{q} < \frac{k}{4}$, then in equilibrium two contracts must be offered, as offering some low contract will not violate the DIC. Now suppose $(1-\theta_l)\underline{q} \geq \frac{k}{4}$. Then the DIC must bind if two contracts are offered. Moreover, $k \in (\frac{2}{3}, 1]$ implies that $u_l^D \leq k/4$. This means that in the duopoly equilibrium, $u_l^d \leq k/4$, i.e., there is no competition for type l. Therefore, the programming problem is the same as before and the first order conditions are given by (6) and (7).

The following lemma summarizes the above analysis.

LEMMA 3 *Both firms offer two contracts in the duopoly equilibrium if (i) $k \leq 1/2$; (ii) $k \in (\frac{1}{2}, \frac{2}{3}]$, and $k \leq \frac{2}{3}\theta_l^2$ or $(1-\theta_l)\underline{q} < \frac{1-k}{2}$; and (iii) $k \in (\frac{2}{3}, 1]$ and $(1-\theta_l)\underline{q} < \frac{k}{4}$. Both firms offer h contract alone in the duopoly equilibrium if $k \in (\frac{2}{3}, 1]$, $(1-\theta_l)\underline{q} \geq \frac{k}{4}$, and either $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k - \frac{1}{2})$ or the* *following two conditions are satisfied:*

$$\alpha(1-\theta_l)\left(\frac{1}{k}-\frac{3}{2}\right)+\frac{1-\alpha}{k}\left[\theta_l^2-\frac{\alpha}{1-\alpha}(\frac{3}{4}k-\frac{1}{2})\right](\theta_l-\underline{q})<0, \tag{8}$$

$$\theta_l\underline{q}-\frac{1}{2}\underline{q}^2<\frac{\alpha}{1-\alpha}\left(\frac{3}{8}k-\frac{1}{4}\right). \tag{9}$$

Proof. See Appendix. ■

2.3 Comparison

In this subsection we compare the duopoly equilibrium with the optimal solution under monopoly. Our focus is on how entry affects the product line, or the number of contracts offered. We first point out that when comparing solutions of only offering an h contract, for $k < 1/2$ the rent to h type under duopoly is higher than that under monopoly ($u_h^D > u_h^{fb}$), while for $k \in (1/2, 1]$ the relationship is reversed ($u_h^D < u_h^{fb}$). The following two propositions identify sufficient conditions under which competition leads to an increase or decrease in number of contracts offered.

PROPOSITION 1 *(Low degree of horizontal differentiation) Suppose $k \leq \frac{1}{2}$. Then competition will never lead to product line pruning, as both firms always offer two contracts in the duopoly equilibrium. Competition may lead to fighting brands. Specifically, if $\frac{k}{2} < (1-\theta_l)\underline{q}$, and either $\theta_l^2 < \frac{k\alpha}{1-\alpha}$ or conditions (4) and (5) are satisfied, then under monopoly it is optimal to offer an h contract only, while under duopoly both firms offer h and l contracts in equilibrium.*

Proof. The results follow from Lemmas 2 and 3. ■

When the degree of horizontal differentiation is low, competition may lead to fighting brands. This is quite intuitive. Given that k is low ($k \leqslant 1/2$), under monopoly, only an h contract is offered (type l is excluded) to reduce the informational rent to type h. On the other hand, under duopoly competition for type h leads to a higher rent to type h. This relaxes the incentive compatibility constraint along the vertical dimension. Hence informational rent consideration becomes less important as type h secures higher rent anyway because of competition. This implies that offering a contract to low-type consumers has less negative impact on the profitability from the high type. As a result, offering a contract to low-type consumers might be profitable, which turns out to be indeed the case when $k \leqslant 1/2$.

Proposition 2 *(High degree of horizontal differentiation). Suppose $k \in (\frac{1}{2}, 1]$. Then competition will never lead to fighting brands, as two contracts are always offered under monopoly. Competition may lead to product line pruning. Specifically, if $k \in (\frac{2}{3}, 1)$, $(1-\theta_l)\underline{q} \geq \frac{k}{4}$, and either $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k - \frac{1}{2})$ or conditions (8) and (9) are satisfied, then under monopoly it is optimal to offer two contracts, while in the duopoly equilibrium each firm offers h contract only.*

Proof. The results follow from Lemmas 2 and 3. ∎

The reason that competition may lead to product line pruning is that besides the competition effect, there is a market share effect by moving from monopoly to duopoly. The competition from the entrant reduces the incumbent's market share for type h, which tends to reduce u_h as there is a smaller market to penetrate into. When k is relatively large, under duopoly neither firm has incentive to steal the other firm's market share for type h, thus entry leads to a lower u_h. This makes the incentive compatibility condition more binding in the vertical dimension. In the mean time, there is a mitigating effect that under duopoly the measure of h-type consumers served by each firm is less than that served under monopoly, which makes the low type relatively more important and firms become more willing to increase the rent to the low type. When the condition $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k - \frac{1}{2})$ is satisfied, the first effect dominates: firms are more concerned about information rent under duopoly. As a result, an incumbent monopolist responds to entry by removing its low quality product targeting type l (the low contract).

Combining Propositions 1 and 2, we see that fighting brands or product line pruning are more likely to occur when the proportion of h type is relatively high and θ_l is close to $\underline{q}$. These conditions ensure that exclusion of the low type is more likely. Moreover, whether entry leads to fighting brands or product line pruning depends on the degree of horizontal differentiation. When the degree of horizontal differentiation is low, under monopoly the market penetration for the high type is high and the rent to the high type is low, which means that the exclusion of the low type is more likely. On the other hand, a low degree of horizontal differentiation means that competition is more intense under duopoly, which leads to higher rent to the high type. This implies that exclusion of the low type is less likely under duopoly. Thus entry might lead to fighting brands. When the degree of horizontal differentiation is high, however, under monopoly a high rent is given to the high type in order to penetrate into the market for the high type, which means that exclusion of the low type is less likely. On the other hand, a high degree of horizontal differentiation means that competition is weak under duopoly. Moreover, the incumbent firm's market share is significantly reduced (to half), which also reduces its incentive to penetrate into the market for the high type. As a result, entry reduces the rent to the high type, and exclusion of the low type becomes more likely, leading to product line pruning.

Suppose that under both monopoly and duopoly two contracts are offered, and let the qualities of the low quality products under monopoly and duopoly be q_l^m and q_l^d, respectively. The following proposition illustrates how entry affects the quality distortion of the low contract.

PROPOSITION 3 *Suppose that two contracts are offered under monopoly and duopoly. If* $k \leq \frac{1}{2}$*, then* $q_l^m \leq q_l^d$.

Proof. See Appendix. ■

The intuition, again, has to do with the effect of competition on the (vertical) DIC constraint and the (horizontal) market shares. When k is small, competition in duopoly leads to higher rent for type h. This relaxes the DIC. On the other hand, the market share for type h becomes smaller in duopoly, which makes each firm more willing to increase the rent to the low type. These effects both contribute to a smaller downward quality distortion.

When $k > 1/2$ and two contracts are offered under both duopoly and monopoly, the quality distortion can be higher or lower under duopoly. This is because with a higher k, competition for the h type under duopoly tends to reduce the rent to the h type. Thus the two effects mentioned early work against each other, leading to ambiguous results regarding quality distortion. The following examples illustrate both possibilities. Suppose $k = 2/3$ and $\theta_l = 1/3$. It can be easily checked that the first-best solution is feasible under monopoly, hence $q_l^m = \theta_l = 1/3$. Under duopoly $u_h^D = 1/6 < 1/4 = \frac{\theta_l^2}{4} + (1 - \theta_l)\theta_l$, thus the first-best solution is not feasible. The binding DIC dictates that $q_l^d < \theta_l = q_l^m$. Next we provide an example in which the quality distortion is higher under monopoly. Suppose $k = 0.54$ and $\theta_l = 0.9$. Since $k \leq \frac{2}{3}\theta_l^2$, we know that the first-best solution is obtained in duopoly and $q_l^d = \theta_l$. However, under monopoly it can be easily verified that the first-best solution is not feasible ($\frac{k}{2} - \frac{\theta_l^2}{4} < (1 - \theta_l)\theta_l$). Thus we conclude that $q_l^m < q_l^d = \theta_l$.

2.4 More Firms

In this subsection we study how further entry affects the product line, or the number of contracts offered by incumbent firms. Denote n as the number of firms. In the horizontal dimension, the location of n firms' products evenly split the unit-length circle. As standard in Salop circular city model, an increase in n in the n-firm model is equivalent to a decrease in k in the duopoly model, as competition may exist only between two adjacent firms (see Yang and Ye, 2008 for a demonstration).[9] For this reason we focus on the comparative statics of the duopoly model with respect to k.

The following proposition shows that the impacts of a decrease in k on the menu of contracts offered depends on the initial level of k.

PROPOSITION 4 *(i) If $k \in (0, \frac{2}{3}]$, then in the duopoly equilibrium a decrease in k can only lead to fighting brands. If two contracts are offered in the duopoly equilibrium under both k and k', where $k > k'$, then $q_l^d(k') \geq q_l^d(k)$. (ii) If $k \in (\frac{2}{3}, 1]$, then in the duopoly equilibrium a decrease in k may result in either fighting brands or product line pruning.*

Proof. See Appendix. ■

In this n-firm model, Proposition 4 implies that whether entry leads to fighting brands or product line pruning depends on the initial degree of competition. When initial competition is fierce (k is small or the initial n is large), then further entry can only lead to fighting brands and a decrease in quality distortion. On the other hand, when initial competition is weak (k is big or the initial n is small), then further entry can lead to either fighting brands or product line pruning. These results again come from the combined competition and market share effects from further entry. When the initial level of competition is high, entry leads to fierce competition for the high type, which increases the rent to the high type. Moreover, entry reduces incumbent firms' effective market share for the high type, which makes the low type more important. These two effects work in the same direction, relax the DIC, and make it potentially profitable to introduce the low contracts. When the initial level of competition is low, the competition effect is absent. But reduced market share for the high type reduces the rent to the high type. This tends to make the DIC constraint more binding. On the other hand, a reduced market share for the high type implies that the low type becomes more

[9] Thus k also captures the competition intensity: the smaller k, the more intense competition between two adjacent firms.

important, which tends to relax the DIC. If the first effect dominates, entry can only lead to the removal of the low contracts. However, if the second effect dominates, entry can only lead to the addition of low contracts.

Our results are consistent with the empirical findings of Seim and Viard (2006), who study how entry into local cellular phone market affects the number of calling plans offered by each incumbent firms.[10] When the initial number of firms is small in a local market, entry reduces the number of calling plans offered by incumbents. However, when the initial number of firms is large, incumbent firms respond to entry by increasing the number of calling plans.

3 Asymmetric Firms

In this section, we study the situation in which the entrant is technologically inferior to the incumbent. In particular, the quality range that the entrant is able to produce is $[\underline{q}, \overline{q}]$, where $\overline{q} < 1$. When the upper bound $\overline{q}$ decreases, we say that the entrant becomes technologically more inferior. We are interested in how a change in $\overline{q}$ affects the incumbent's response to entry. To simplify our analysis, we assume that $\overline{q} > \theta_l$. This assumption implies that, for the entrant, the first-best quality for type h is always strictly higher than that for type l. As a result, Lemma 1 still applies, which means that the entrant will either offer two contracts or offer h contract targeting at type h only.

In terms of the menu of contracts offered, there are four possible outcomes: both firms offer high contracts only, both firms offer two contracts, the incumbent offers two contracts while the entrant offers only a high contract, and the incumbent offers only a high contract while the entrant offers two contracts. The case where the entrant only offers a low contract cannot happen because high types not covered by the incumbent would take entrant's low contract, leading to pooling,[11] which is not possible.

[10] Other empirical analysis of the early US cellular phone industry can be found in, e.g., Miravete and Röller (2004) and Miravete (2009).

[11] The only situation where the entrant only offers low contract is when the incumbent covers all the high type, but this could be included in our previous four cases.

3.1 Full Information

We first specify the first-best solution under full information. We use superscripts I and E to denote the incumbent and entrant, respectively, and that u_h^I and u_h^E are the utilities offered to type h by the incumbent and entrant, respectively. Note that $q_h^I = 1$ and $q_h^E = \overline{q}$. We need to discuss two cases. First, when $u_h^I + u_h^E \leq k/2$ (the market is not fully covered), the problem can be written as follows:

$$\text{I} : \max_{u_h^I} \alpha \frac{u_h^I}{k} \left(\frac{1}{2} - u_h^I \right)$$

and

$$\text{E} : \max_{u_h^E} \alpha \frac{u_h^E}{k} \left(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^E \right)$$

Second, when $u_h^I + u_h^E \geq k/2$ (the market is fully covered), the problem becomes:

$$\text{I} : \max_{u_h^I} \alpha \left(\frac{1}{4} + \frac{u_h^I - u_h^E}{2k} \right) \left(\frac{1}{2} - u_h^I \right)$$

and

$$\text{E} : \max_{u_h^E} \alpha \left(\frac{1}{4} + \frac{u_h^E - u_h^I}{2k} \right) \left(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^E \right)$$

The equilibrium under full information is given below:

$$(u_h^I, u_h^E) = \begin{cases} \left(\frac{1-k}{2} - \frac{1}{6}(1-\overline{q})^2, \frac{1-k}{2} - \frac{1}{3}(1-\overline{q})^2\right) & \text{if} \quad k \leq \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]; \\ \left(\frac{k}{2}\frac{1}{2-(1-\overline{q})^2}, \frac{k}{2}\frac{1-(1-\overline{q})^2}{2-(1-\overline{q})^2}\right) & \text{if} \quad \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2] < k \leq 1 - \frac{1}{2}(1-\overline{q})^2; \\ \left(\frac{1}{4}, \frac{\overline{q}}{2} - \frac{\overline{q}^2}{4}\right) & \text{if} \quad k > 1 - \frac{1}{2}(1-\overline{q})^2. \end{cases}$$

When $\overline{q} = 1$, the equilibrium characterized above is consistent with the previously derived u_h^D, the equilibrium with symmetric firms. To ensure interaction between the incumbent and the entrant, we assume that $k \leq 1 - \frac{1}{2}(1-\overline{q})^2$. Note that when $k \in (\frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2], 1 - \frac{1}{2}(1-\overline{q})^2]$, although the market is fully covered, there is no effective competition for the high type.

The full-information low contracts are the same as those in the symmetric case (as the competition between low contracts remains symmetric). Specifically, $q_l^I = q_l^E = \theta_l$, and

$$u_l^D = \begin{cases} \frac{\theta_l^2}{4} & \text{if } k \in [\theta_l^2, 1] \\ \frac{k}{4} & \text{if } k \in [\frac{2}{3}\theta_l^2, \theta_l^2) \\ \frac{\theta_l^2 - k}{2} & \text{if } k \in (0, \frac{2}{3}\theta_l^2) \end{cases} .$$

3.2 Private Information

Now we turn our attention to the case of private information. We first identify a sufficient condition under which both firms offer two contracts in the duopoly equilibrium.

LEMMA 4 *If* $k < \frac{1}{2} - \frac{1}{3}(1-\overline{q})^2 \equiv \widehat{k}_1(\overline{q})$, *in the duopoly equilibrium the incumbent offers two contracts. If* $k < \frac{1}{2} - \frac{2}{3}(1-\overline{q})^2 \equiv \widehat{k}_2(\overline{q})$, *in the duopoly equilibrium both firms offer two contracts.*

Proof. See Appendix. ■

Recall that in the duopoly equilibrium with symmetric firms, the incumbent will offer two contracts if $k \leq 1/2$. Note that $\widehat{k}_1(\overline{q}) < 1/2$. Moreover, $\widehat{k}_1(\overline{q})$ is increasing in $\overline{q}$. This suggests that when the entrant becomes more inferior, the condition under which the incumbent will offer two contracts becomes more stringent. In later examples, we will show that when $\widehat{k}_1(\overline{q}) < k < 1/2$, the incumbent might offer h contract only with asymmetric firms. Under private information we use superscripts i and e to denote the incumbent and entrant, respectively.

PROPOSITION 5 *In the duopoly equilibrium, (i) if both firms offer two contracts, then the quality distortion of the incumbent is smaller,* $q_l^i \geq q_l^e$; *(ii) if the entrant offers two contracts, then the incumbent must also offer two contracts.*

Proof. See Appendix. ■

Part (i) of Proposition 5 tells us that, when both firms offer two contracts, the low quality of the incumbent must be higher than that of the entrant. This result comes from the fact that the entrant is technologically inferior. In equilibrium the incumbent will offer higher rent to the high type than the entrant does. This means that the DIC is more relaxed for the incumbent than for the entrant. If both firms offer two contracts, the quality distortion for the incumbent's low contract must be smaller. For the same reason, if the entrant has an incentive to offer the low contract, then the incumbent must offer two contracts as well, leading to part (ii) of Proposition 5. Note that the result $q_l^i \geq q_l^e$ is consistent with Johnson and Myatt (2003), who show that the incumbent will never offer products that are of quality lower than that of the entrant's lowest-quality product.

Part (ii) of Proposition 5 rules out the equilibrium where the incumbent offers only an h contract and the entrant offers both contracts. The following example shows that in equilibrium it is possible for the incumbent to offer two contracts and the entrant to offer only an h contract. Consider the following case: $k = 1/2$, $\theta_l = 0.25$, $\underline{q} = 0.2$, $\overline{q} = 0.3$ and $\alpha = 3/4$. In this case, $u_h^I = 0.168$, $u_h^E = 0.067$, and $(1-\theta_l)\underline{q} = 0.15$. Since $u_h^I > (1-\theta_l)\underline{q}$, the incumbent must offer two contracts under private information. Now we claim that in equilibrium the incumbent offers two contracts with $u_h^i = u_h^I$ and the entrant only offers an h contract with $u_h^e = u_h^E$. From the incumbent's FOC (13) (in the appendix), it is not difficult to see that the LHS (excluding μ^i) is negative if we impose $u_l^i = 0$. Therefore, $u_h^i = u_h^I$. Now inspect the entrant's FOC (15) (also in the appendix) with $u_h^e = 0.15$ and $u_h^i = 0.168$. It can be verified that the LHS (excluding μ^e) is negative if we impose $u_l^e = 0$, which means that it is not profitable for the entrant to offer the low contract. Thus in this example we show that the incumbent will offer two contracts while the entrant only offers one contract.

PROPOSITION 6 *Suppose $k \leq \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$. Let $\overline{q}' > \overline{q}$. If in the duopoly equilibrium with $\overline{q}$ the incumbent offers two contracts, then in the duopoly equilibrium with $\overline{q}'$ the incumbent must offer two contracts as well. Moreover, $q_l^{i\prime} \geq q_l^i$.*

Proof. See Appendix. ■

The intuition for Proposition 6 is as follows. When k is so small that there is effective competition for the high type, as the entrant becomes less inferior, the competition for the high type becomes more fierce. This leads to two effects. First, the incumbent will offer higher rent to the high type due to increased competition, and the DIC is slackened. Second, the incumbent's market share for the high type becomes smaller. This makes the low type relatively more important for the incumbent and offering a low contract more likely. Both effects increase the incentive for the incumbent to introduce a low contract. If the low contract has been originally offered, the quality for the low contract will be less distorted.

An implication from Proposition 6 is that when k is sufficiently small (so that there is effective competition for the high type), as the entrant becomes more inferior, fighting brands become less likely and product pruning becomes more likely. The following example shows that a decrease in $\overline{q}$ makes product pruning more likely. Let $k = 0.5$, $\theta_l = 0.38$, $\underline{q} = 0.35$, and $\alpha = 3/4$. By Lemma 2, in monopoly the incumbent offers two contracts. If $\overline{q} = 1$, by Lemma 3, under the duopoly equilibrium the incumbent offers two contracts as well. Now suppose $\overline{q} = 0.4$. Note that $u_h^I = 0.19$ and $u_h^E = 0.13$. We argue that in the duopoly equilibrium both firms only offer an h contract, with $u_h^i = u_h^I$ and $u_h^e = u_h^E$. To show this, first note that $(1-\theta_l)\underline{q} = 0.217 > u_h^I$. Therefore, if two contracts are offered, the DIC must bind for both the incumbent and entrant. We first check the incumbent's incentive given $u_h^e = 0.13$. If the incumbent offers two contracts, then $u_h^i \geq 0.217$. In that case it can be easily verified that the LHS of (13) is less than zero. Thus no u_l^i satisfies (13), and the incumbent has no incentive to offer a low contract. Now consider the entrant's incentive given $u_h^i = 0.19$. When the entrant offers two contracts, then $u_h^e \geq 0.217$, in which case it can be verified that the LHS of (15) is less than zero. Therefore, no u_l^i satisfies (13), and the entrant has no incentive to offer a low contract either. This example thus suggests that when the entrant is very inferior, entry leads to product line pruning.

Note that a similar result does not hold when k is relatively large such that there is no effective competition for the high type. In that case, when the entrant becomes less inferior there are two opposing effects. First, the incumbent tends to reduce its rent to the high type. This can be seen from the fact that u_h^I is decreasing in $\overline{q}$ when $k > \frac{2}{3}[1-\frac{1}{2}(1-\overline{q})^2]$. This effect tends to make the DIC more stringent. On the other hand, an increase in $\overline{q}$ reduces the incumbent's market share for the high type. This makes the low type relatively more important and tends to relax the DIC. Whether the incumbent' DIC is slackened or not as $\overline{q}$ increases depends on which effect dominates.

4 Three-type Model with Partial Pooling

In this section we consider a model with three vertical types. Suppose in the vertical dimension consumers have three types: θ_h, θ_m, and θ_l, where $\theta_h = 1 > \theta_m > \theta_l$. The proportions of types are α_h, α_m, and α_l, respectively ($\alpha_h + \alpha_m + \alpha_l = 1$). All the other assumptions are the same as in the base model.

As in the two-type model, in the three-type model entry might lead to fighting brands or product pruning. Since three contracts can be potentially offered, entry may lead to the introduction (or removal) of a middle quality product, a low quality product, or both (contracts). Although the pattern can be more complicated, they are qualitatively the same as fighting brand and product line pruning in the two-type model, since expansion or contraction of the product lines (contracts) only occurs at the low end. With three types, since pooling of the middle and the low types becomes a possibility, the expansion or contraction of the set of contracts offered might occur for the middle product (contract), such as Garmin's response to entry by releasing the Quest series or Microsoft's response to entry by releasing Office 2003. This new feature will be the focus of this section.

4.1 Entry leads to the introduction of a middle contract

Monopoly. We start with the analysis of monopoly. Under monopoly, the full information solution is as follows: $q_i^{fb} = \theta_i$ and

$$u_i^{fb} = \begin{cases} \frac{\theta_i^2}{4} & \text{if } k \in (\frac{\theta_i^2}{2}, 1] \\ \frac{k}{2} & \text{if } k \in (0, \frac{\theta_i^2}{2}] \end{cases} .$$

Under private information, we again have $q_h = 1$. Similarly to the argument in Lemma 1, we can show that type h is never pooled with other two types. Overall, we have four cases to consider: only an h contract is offered, only h and m contracts are offered and type l is excluded, three contracts are offered (full separating), and two contracts are offered, with types m and l pooling at the low contract (partial pooling). We are interested in the last case, as it is qualitatively different from the two-type base model. The relevant ICs are: $u_h - u_m \geq (1-\theta_m)q_m$ (DIC$_{hm}$), $u_m - u_l \geq (\theta_m - \theta_l)q_l$ (DIC$_{ml}$), $u_h - u_l \geq (1-\theta_l)q_l$ (DIC$_{hl}$), and $u_m - u_l \leq (\theta_m - \theta_l)q_m$ (UIC$_{lm}$). Note that when $q_m \geq q_l$, then DIC$_{hl}$ is redundant.

Lemma 5 *Suppose $k \leq 1/2$. If $\theta_m < 3\theta_l$, $\frac{k}{2} < \frac{\theta_m^2}{4} + (1-\theta_m)\theta_m$, $\alpha_h(1-\theta_m) \geq \alpha_m(\theta_m - \underline{q})$, $\alpha_h(1-\theta_l) \geq \frac{\alpha_m}{k}(\theta_m - \theta_l)\theta_l^2 + (1-\alpha_h)(\theta_l - \underline{q})$, $\theta_m \underline{q} - \frac{1}{2}\underline{q}^2 \leq \frac{\alpha_h}{\alpha_m}\frac{k}{2}$, and the following condition*

$$(1-\alpha_h)\frac{(\theta_l \underline{q} - \frac{1}{2}\underline{q}^2)^2 - (\frac{\alpha_h}{1-\alpha_h}\frac{k}{2})^2}{4k} - \frac{\alpha_h}{2}\left[\frac{1}{2}\left(\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 - \frac{\alpha_h}{1-\alpha_h}\frac{k}{2}\right) + (1-\theta_l)\underline{q} - \frac{k}{2}\right] > 0$$

hold, then under monopoly the optimal menu of contracts exhibits partial pooling: two contracts are offered, with the high contract targeting at type h, and types m and l pooled at the low contract with $q_l = \underline{q}$.

Partial pooling is likely to occur when the proportion of the middle type (α_m) is small, the proportion of the high type (α_h) is also small relative to that of the low type (α_l), and $\underline{q}$ is fairly close to θ_l. A relatively large α_l gives the firm an incentive to cover the low type, and that α_m and $\underline{q}$ being small (close to θ_l) implies that separating type m from type l is costly, which gives rise to partial pooling in the optimal contract.

Duopoly. Under full information, the duopoly equilibrium contracts take the following form: $q_i^D = \theta_i$, and

$$u_i^D = \begin{cases} \frac{\theta_i^2}{4} & \text{if } k \in [\theta_i^2, 1] \\ \frac{k}{4} & \text{if } k \in [\frac{2}{3}\theta_i^2, \theta_i^2) \\ \frac{\theta_i^2 - k}{2} & \text{if } k \in (0, \frac{2}{3}\theta_i^2) \end{cases}.$$

We are interested in the case $k \leq 1/2$, and will focus on the duopoly equilibrium in which the menu of contracts is fully separating.

Given that $k \leq 1/2$, $u_h^D = \frac{1-k}{2}$. Since $(1-\theta_i)\underline{q} < 1/4 \leq \frac{1-k}{2}$, in the duopoly equilibrium at least two contracts are offered. One sufficient condition to guarantee full separation is that $k \leq \frac{2}{3}\theta_l^2$. In this case, competition exists for all three types. The full information solution always satisfies the DICs: for $\theta_i > \theta_j$,

$$u_i^D - u_j^D = \frac{\theta_i^2 - \theta_j^2}{2} > (\theta_i - \theta_j)\theta_j.$$

Therefore, the duopoly equilibrium exhibits full separation and no quality distortion.

We next identify another sufficient condition. Suppose that $k \in [\frac{2}{3}\theta_m^2, \theta_m^2)$, and $2 - 3k \geq 4(1-\theta_m)\theta_m$. By the first condition, $u_m^D = k/4$. By the second condition, the DIC$_{hm}$ is slack in the full-information solution. Therefore, in the duopoly equilibrium we must have $q_m^d = \theta_m$ (no quality distortion for type m). We further assume that $(\theta_m - \theta_l)\underline{q} < \frac{k}{4}$. This condition implies that if q_l is low enough, offering a contract to type l will not affect the DIC$_{ml}$. Therefore, in the duopoly equilibrium, each firm must offer three contracts (fully separating). The following lemma summarizes the results of duopoly.

Lemma 6 *Suppose $k \leq 1/2$. (i) If $k \leq \frac{2}{3}\theta_l^2$, then in the duopoly equilibrium the full-information solution is feasible: each firm offers three contracts without quality distortion. (ii) If $k \in [\frac{2}{3}\theta_m^2, \theta_m^2)$, $2 - 3k \geq 4(1-\theta_m)\theta_m$, and $(\theta_m - \theta_l)\underline{q} < \frac{k}{4}$, then in the duopoly equilibrium each firm offers three contracts, with $q_m^d = \theta_m$.*

Combining Lemma 5 and Lemma 6, we have the following result.

Proposition 7 *Let $k \leq 1/2$. If the parameter values are such that all the conditions in Lemma 5 are satisfied, and either $k \leq \frac{2}{3}\theta_l^2$ or the conditions in part (ii) of Lemma 6 are satisfied, then under monopoly two contracts are offered, with the low and middle types pooled at $\underline{q}$, while in the duopoly equilibrium each firm offers three contracts (fully separating).*

It is easy to see that there are parameter values such that both conditions in Lemma 5 and Lemma 6 are satisfied. This is because the conditions in Lemma 6 have nothing to do with the distribution of types. So we can choose α's freely to satisfy the conditions in Lemma 5.[12]

Proposition 7 illustrates that entry can expand the incumbent's menu of contracts by converting a partial pooling equilibrium to a fully separating equilibrium. We should emphasize that this scenario is different from fighting brands. Recall that in the case of fighting brands, entry leads to an introduction of a low quality good (contract). However, in the scenario described by Proposition 7, the low quality good (contract) is offered under monopoly, and entry leads to an introduction of a middle quality good (contract). Our analysis thus suggests a new pattern of product line expansion that is different from fighting brands. Such a pattern is consistent with, for example, a finding in Seim and Viard (2009) that with more entry, firms may spread their calling plans more evenly over the usage spectrum.

The driving force behind Proposition 7 is again the interaction between horizontal competition and vertical screening. When k is small, competition for high types after entry leads to higher rent to high types. This relaxes the sorting constraint and makes informational rent consideration along the vertical dimension less important. As a result, the incumbent has less incentive to exclude low types or to pool the low types.

4.2 Entry leads to the removal of a middle contract

In this subsection we provide an analysis of the opposite case, which exhibits fully separating under monopoly but partial pooling under duopoly. In effect we will identify conditions under which entry will lead to fewer contracts offered. We restrict attention to the case that $\frac{1}{2} < k < \frac{2}{3}$.

Monopoly. Under monopoly, $u_h^{fb} = \frac{1}{4}$, $u_m^{fb} = \frac{\theta_m^2}{4}$, and $u_l^{fb} = \frac{\theta_l^2}{4}$. When $u_m^{fb} + (1-\theta_m)\theta_m \leq u_h^{fb}$, the DIC$_{hm}$ is slack under full information. When $u_m^{fb} > (\theta_m - \theta_l)\underline{q}$, which is always valid since $\theta_l > \underline{q}$, it is always profitable to offer a low contract. Overall, we conclude that if $\theta_m \leq \frac{1}{3}$, it is optimal to offer three separate contracts under monopoly.

[12] The following parameter values satisfy all the conditions in Lemma 5 and the conditions in part (ii) of Lemma 6: $k = 0.3$, $\theta_m = 0.6$, $\theta_l = 0.5$, $\underline{q} = 0.4$, $\alpha_h = 0.19$, $\alpha_m = 0.1$, $\alpha_l = 0.71$.

Duopoly. Under duopoly, $u_h^D = \frac{1-k}{2}$, $u_m^D = \frac{\theta_m^2}{4}$, and $u_l^D = \frac{\theta_l^2}{4}$.[13] When $u_m^D + (1-\theta_m)\theta_m > u_h^D$, DIC_{hm} binds; when $u_m^D < u_l^D + (\theta_m - \theta_l)\theta_l$, the DIC_{ml} also binds. Combining these two conditions, we have that if $(\theta_m - \frac{2}{3})^2 < \frac{6k-2}{9}$ and $\theta_m < 3\theta_l$, both DICs bind.

PROPOSITION 8 *When $\frac{1}{2} < k < \frac{2}{3}$, $\theta_m \leq \frac{1}{3}$, and $\theta_m < 3\theta_l$, if conditions (30), (31), (35), and (36) (listed in the appendix) hold, then the firm will offer three separate contracts under monopoly while partial pooling of middle and low types would take place under duopoly.*

Proof. See Appendix. ■

Proposition 8 shows that when k is relatively large, α_m is sufficiently small, and type m and type l are fairly close to each other but rather far away from type h, then the incumbent monopolist responds to entry by removing the middle contract. The rough intuition is as follows. A relatively large k makes the monopolist willing to give the high type a high rent in order to penetrate into its market. This means that the IC constraints in the vertical dimension are fairly relaxed, leading to a fully separating equilibrium under monopoly. On the other hand, a relatively large k under duopoly leads to a lower rent to the high type, which makes the IC constraints in the vertical dimension more stringent. Given that α_m is sufficiently small, and type m and type l are fairly close to each other, entry makes it too costly for firms to offer a separate contract to the middle type, thus the middle contract of the incumbent is removed. We can easily choose α_h to satisfy the conditions in the proposition. One such choice is the following combination of parameters: $k = 0.62$, $\theta_h = 1$, $\theta_m = 0.33$, $\theta_l = 0.32$, $\underline{q} = 0.31$, and $\alpha_h = 0.62$, $\alpha_m = 0.10$, $\alpha_l = 0.28$.

The practice of removing some middle contracts (or middle-ranged quality product line) in response to entry is very common. For example, following the entry of Toyota into North American market, Buick reduced the number of its mid-size models offered from two (Special and Skylark) to one (Skylark) in 1970. Ford also reduced its mid-size model line from two models, Fairlane and Torino, to just one model, Torino. In response to the entry of Honda in 1980, Ford cut LTD II from its mid-size car line.

[13] Given that $\theta_m \leq \frac{1}{3}$ and $k \geq 1/2$, $\theta_m^2 < k$. This implies that both the low and the middle types will not be fully covered in the horizontal dimension.

5 Conclusion

In this research, we study how entry or increased competition affects the product line or the variety of contracts offered in a standard Salop circular city model with both horizontally and vertically differentiated products. Our analysis offers a novel explanation for the use of fighting brands and product line pruning. Our main result is that when the degree of horizontal differentiation is low or the horizontal competition level is keen, entry will typically lead to fighting brands under certain conditions; when the degree of horizontal differentiation is high or the horizontal competition level is weak, however, entry will typically lead to product line pruning under certain conditions. The conditions we identify are intuitive, as they can all be explained by the interactions between horizontal differentiation (competition) and vertical screening.

The extension to asymmetric firms and three (vertical) types of consumers further confirm the general insights obtained from our base model. In particular, our analysis of three type models reveals an interesting pattern between fully separating and partial pooling equilibria, and offers an explanation for why incumbent firms may adjust middle range of product line (middle contracts) in response to competition, which is different from fighting brands or product line pruning. Our result does point to some more subtle effects of entry or increased competition on the product line or the variety of contracts offered.

From both theoretical and practical points of view, it would be desirable to work out a more general model allowing for any finite number of types. However, doing so presents some technical difficulty, as the incentive comparability constraints along the vertical dimension will become quite complicated. While we believe that the main insights obtained from our current model is quite robust, we should try to generalize our analysis in future research.

Appendix

Proof of Lemma 1: Let t be the transfer under the single contract. First consider the case $q \in [\underline{q}, 1)$. Suppose the monopolist introduces another contract targeting at type h: $q_h = 1$ and $t_h = t + (1 - q)$. By construction, it can be verified that $u_h(q,t) = q - t = q_h - t_h = u_h(q_h, t_h)$. Thus type h will accept contract h and its market coverage does not change. On the other hand, $u_l(q,t) = \theta_l q - t > \theta_l q_h - t_h = u_l(q_h, t_h)$. Hence type l will still buy the original contract and the firm's profit from type l does not change. However, the profit per consumer from type h increases under contract h: under the original contract the profit margin is $t - \frac{1}{2}q^2$, and under contract h it becomes $t + (1 - q) - \frac{1}{2}$, which is strictly greater than $t - \frac{1}{2}q^2$ since $q < 1$. Because the market share for type h remains the same, the introduction of contract h strictly increases the firm's profit.

Next consider the case $q = 1$. Suppose the monopolist introduces another contract targeting at type l: $q_l = \theta_l$ and $t_l = t - \theta_l(1 - \theta_l)$. By construction, type l is indifferent between the original contract and contract l. Thus type l selects the l contract and the market share for type l does not change. It can be verified that type h prefers the original contract: $(1-t) - (\theta_l - t_l) = (1-\theta_l)^2 > 0$. Thus type h will stick to the old contract and the profit from type h agents does not change. However, the profit margin from type l becomes higher:

$$\left(t_l - \frac{1}{2}\theta_l^2\right) - \left(t - \frac{1}{2}\right) = \frac{1}{2}\theta_l^2 - \theta_l + \frac{1}{2} = \frac{1}{2}(1-\theta_l)^2 > 0.$$

Therefore, the introduction of contract l strictly raises the firm's profit. ■

Proof of Lemma 3: We only need to prove the last part of the result, as the rest have been shown in the previous analysis. Suppose two contracts are offered for both firms. Consider the first order condition (6) that characterizes the symmetric equilibrium. Note that in the LHS of condition (6), $\frac{1}{2k}(\frac{1}{2} - u_h^d) - \frac{1}{4} \leq \frac{1}{4k} - \frac{3}{8}$ since $u_h^d \geq u_h^D = \frac{k}{4}$. Given that $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k - \frac{1}{2})$, we have

$$\begin{aligned}
\text{LHS of (6)} &\leq \alpha\left(\frac{1}{4k} - \frac{3}{8}\right) + \frac{1-\alpha}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - 2u_l\right) \\
&\leq \alpha\left(\frac{1}{4k} - \frac{3}{8}\right) + \frac{1-\alpha}{2k}\theta_l^2 \\
&< 0.
\end{aligned}$$

This implies that $\mu_D > 0$ and $u_l^d = 0$. Therefore, in equilibrium both firms must offer h contract alone, with $u_h^d = u_h^D = \frac{k}{4}$. Now consider conditions (8) and (9). Condition (8) ensures that the LHS of (7) is negative, thus $q_l^d = \underline{q}$. With $q_l^d = \underline{q}$, condition (9) ensures that the LHS of (6) is negative, thus $u_l^d = 0$. ■

Proof of Proposition 3: First we show that if the DIC does not bind (the full-information solution is feasible) under monopoly, then it does not bind under duopoly either. From the previous analysis, when $k \leq 1/2$, the DIC does not bind under monopoly if and only if

$$\frac{k}{2} - \frac{\theta_l^2}{4} \geq (1-\theta_l)\theta_l. \tag{10}$$

On the other hand, when $k \leq 1/2$, the DIC does not bind under duopoly if and only if one of the following three conditions hold: (i) $k \in (0, \frac{2}{3}\theta_l^2)$,

$$\text{(ii) } \frac{1-k}{2} - \frac{k}{4} \geq (1-\theta_l)\theta_l \text{ if } k \in [\frac{2}{3}\theta_l^2, \theta_l^2), \text{ and (iii) } \frac{1-k}{2} - \frac{\theta_l^2}{4} \geq (1-\theta_l)\theta_l \text{ if } k \in [\theta_l^2, 1]. \tag{11}$$

Comparing (10) and (11), we see that if (10) is satisfied then (11) must be satisfied. This result implies that whenever the DIC is slack under duopoly, we have $q_l^m \leq q_l^d$.

What remains to be shown is that $q_l^m \leq q_l^d$ when the DIC binds under both monopoly and duopoly. If $q_l^m = \underline{q}$, then $q_l^d \geq q_l^m$ holds trivially. So we focus on the case that $q_l^m > \underline{q}$. Let the optimal solution under monopoly be (q_l^m, u_l^m). Suppose $(q_l^d, u_l^d) = (q_l^m, u_l^m)$. We will Compare the LHS of the first order conditions (2) and (6) with μs being 0. Since $u_h^d \leq 1/2$ (the maximum social surplus of the high type), the first term in (2) is strictly less than that in (6). Given that (2) holds, the LHS of (6) must be strictly positive when $(q_l^d, u_l^d) = (q_l^m, u_l^m)$. By the same procedure, we can show that the LHS of (7) is strictly positive when $(q_l^d, u_l^d) = (q_l^m, u_l^m)$. This means that each firm can increase its profit by offering $(q_l^d, u_l^d) > (q_l^m, u_l^m)$. This proves that $q_l^m \leq q_l^d$ when DIC binds under both monopoly and duopoly. ■

Proof of Proposition 4: First consider the case $k \in (0, \frac{2}{3}]$. Note that the full-information u_h^D is increasing in k. Moreover, as k becomes smaller the DIC is less likely to bind. Therefore, what remains to be shown is that the result of part (i) holds when the DIC binds under both k and k'. Suppose two contracts are offered in the duopoly equilibrium under k. Then it must be the case that either (a) $(1-\theta_l)\underline{q} < \frac{1-k}{2}$, or (b) $(1-\theta_l)\underline{q} \geq \frac{1-k}{2}$,and (6) and (7) have a solution $(q_l^d(k), u_l^d(k))$ with $q_l^d(k) \in [\underline{q}, \theta_l]$ and $u_l^d(k) > 0$. Consider case (a). Since $k' < k$, we also have $(1-\theta_l)\underline{q} < \frac{1-k'}{2}$.

Therefore, two contracts must be offered in the duopoly equilibrium under k'. Now consider case (b). Substituting $q_l^d(k)$ and $u_l^d(k)$ into (6) and (7) under k', we have that the LHS of both (6) and (7) are strictly greater than 0, which is the LHS of FOC's under k. Therefore, (6) and (7) have a solution $(q_l^d(k'), u_l^d(k'))$ with $q_l^d(k') \geq q_l^d(k) \geq \underline{q}$ and $u_l^d(k') > u_l^d(k) > 0$, hence in the duopoly equilibrium under k', two contracts must be offered and the quality distortion decreases. This proves part (i).

To show part (ii), we find two examples in which a decrease in k leads to fighting brands and product line pruning, respectively. Suppose $k' \geq 2/3$. First, we provide an example in which product line pruning occurs. Consider the parameter space such that the following conditions hold: $\frac{k'}{4} \leq (1-\theta_l)\underline{q} < \frac{k}{4}$ and $\theta_l^2 < \frac{\alpha}{1-\alpha}(\frac{3}{4}k' - \frac{1}{2})$. Then by part (iii) of Lemma 3, in the duopoly equilibrium under k both contracts are offered and in the duopoly equilibrium under k' only the h contract is offered. Thus a decrease in k leads to product line pruning. Next, we provide an example in which the number of contracts increases. Consider the parameter space such that the following conditions hold: $(1-\theta_l)\underline{q} \geq \frac{k}{4}$, $\theta_l^2 \leq \frac{\alpha}{1-\alpha}(\frac{3}{4}k - \frac{1}{2})$, and

$$\theta_l \underline{q} - \frac{1}{2}\underline{q}^2 > \frac{\alpha}{1-\alpha}(\frac{k}{8} + \frac{k'}{4} - \frac{1}{4}). \tag{12}$$

By part (iii) of Lemma 3, the first two conditions ensure that in the duopoly equilibrium under k only the h contract is offered. Now consider the LHS of (6) under k'. Condition (12) implies that when $q_l = \underline{q}$ and $u_h^d = \frac{k}{4} > \frac{k'}{4} = u_h^D(k')$, the LHS is strictly greater than 0. Therefore, under k' the equations (6) and (7) have a solution with $u_l^d > 0$. Hence two contracts must be offered in the duopoly equilibrium. Thus a decrease in k leads to fighting brands. ■

Proof of Lemma 4: First, note that $k < \widehat{k}_2(\overline{q})$ implies that $k < \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$ (there is competition for type h). If $k < \widehat{k}_1(\overline{q})$, we have

$$u_h^I = \frac{1-k}{2} - \frac{1}{6}(1-\overline{q})^2 > \frac{1}{4} > (1-\theta_l)\underline{q}.$$

The above inequality means that the incumbent must offer two contracts. Now suppose $k < \widehat{k}_2(\overline{q})$, then we have

$$u_h^I > u_h^E = \frac{1-k}{2} - \frac{1}{3}(1-\overline{q})^2 > \frac{1}{4} > (1-\theta_l)\underline{q}.$$

The above inequalities imply that two firms must offer two contracts in the duopoly equilibrium. ■

Proof of Proposition 5: From the solutions to the full information benchmark, we observe that $u_h^I > u_h^E$ and $u_l^I = u_l^E$. Therefore, regarding whether the unconstrained solution is feasible, we have three possible scenarios. (1) The unconstrained solutions are feasible both for the incumbent and the entrant. In this case, we have $q_l^i = q_l^e = \theta_l$. (2) The unconstrained solution is feasible for the incumbent, but not feasible for the entrant. In this case, we have $q_l^i = \theta_l > q_l^e$. (3) The unconstrained solution is not feasible for both the incumbent and the entrant. In this case, the DIC must be binding for both firms. We inspect case (3) in more detail.

In case (3), we must have $k > \frac{2}{3}\theta_l^2$. This is because if $k \leq \frac{2}{3}\theta_l^2$, then $k < \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$, and the unconstrained solution is feasible for the incumbent. Given that $k > \frac{2}{3}\theta_l^2$, there is no effective competition for the low type. As a result, the programming problem becomes:

$$\text{I:} \quad \max_{u_l^i, q_l^i} \quad \alpha\left[\frac{1}{4} + \frac{u_l^i - (1-\theta_l)q_l^i - u_h^e}{2k}\right]\left(\frac{1}{2} - u_h^i\right) + (1-\alpha)\frac{u_l^i}{k}\left[\theta_l q_l^i - \frac{1}{2}(q_l^i)^2 - u_l^i\right]$$

$$\text{s.t.} \quad q_l^i \geq \underline{q},\ u_l^i \geq 0$$

and

$$\text{E:} \quad \max_{u_l^e, q_l^e} \quad \alpha\left[\frac{1}{4} + \frac{u_l^e - (1-\theta_l)q_l^e - u_h^i}{2k}\right]\left[\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^e\right] + (1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^e - \frac{1}{2}(q_l^e)^2 - u_l^e\right]$$

$$\text{s.t.} \quad \overline{q} > q_l^e \geq \underline{q},\ u_l^e \geq 0$$

The FOC's for I are:

$$\frac{\alpha}{2k}\left(\frac{1}{2} - u_l^i - (1-\theta_l)q_l^i\right) - \alpha\left[\frac{1}{4} + \frac{u_l^i + (1-\theta_l)q_l^i - u_h^e}{2k}\right] + \frac{1-\alpha}{k}\left[\theta_l q_l^i - \frac{1}{2}(q_l^i)^2 - 2u_l^i\right] + \mu^i = 0 \tag{13}$$

$$\frac{\alpha(1-\theta_l)}{2k}\left(\frac{1}{2} - u_l^i - (1-\theta_l)q_l^i\right) - \alpha(1-\theta_l)\left[\frac{1}{4} + \frac{u_l^i + (1-\theta_l)q_l^i - u_h^e}{2k}\right] + (1-\alpha)\frac{u_l^i}{k}(\theta_l - q_l^i) + \lambda^i = 0 \tag{14}$$

and FOC's for E are:

$$\frac{\alpha}{2k}\left[\overline{q} - \frac{1}{2}(\overline{q})^2 - u_l^e - (1-\theta_l)q_l^e\right] - \alpha\left[\frac{1}{4} + \frac{u_l^e + (1-\theta_l)q_l^e - u_h^i}{2k}\right] + \frac{1-\alpha}{k}\left[\theta_l q_l^e - \frac{1}{2}(q_l^e)^2 - u_l^e\right] - \frac{(1-\alpha)}{k}u_l^e + \mu^e = 0 \tag{15}$$

$$\frac{\alpha(1-\theta_l)}{2k}\left[\overline{q} - \frac{1}{2}(\overline{q})^2 - u_l^e - (1-\theta_l^e)q_l^e\right] - \alpha(1-\theta_l)\left[\frac{1}{4} + \frac{u_l^e + (1-\theta_l)q_l^e - u_h^i}{2k}\right] + \frac{(1-\alpha)}{k}u_l^e(\theta_l - q_l^e) + \lambda^e = 0 \tag{16}$$

where μ^i and λ^i are Lagrangian multipliers for the constraints in I's problem, and μ^e and λ^e are Lagrangian multipliers for the constraints in E's problem.

Let (u_l^e, q_l^e) be the solution to the above problem for the entrant. Given that the entrant offers two contracts, $u_l^e > 0$ thus $\mu^e = 0$. If $\lambda^e > 0$, then $q_l^e = \underline{q}$. Since the incumbent also offers two contracts, we immediately have $q_l^i \geq q_l^e$. Now suppose that $\lambda^e = 0$. We demonstrate that it must be the case that $(u_l^i, q_l^i) \geq (u_l^e, q_l^e)$. Suppose $(u_l^i, q_l^i) = (u_l^e, q_l^e)$. Then, since $\frac{1}{2} > \overline{q} - \frac{1}{2}\overline{q}^2$, the LHS of (13) is strictly greater than that of (15), which is zero, and the LHS of (14) is strictly greater than that of (16), which is zero. This implies that the incumbent would have incentive to raise either u_l^i, or q^i, or both. Therefore, $(u_l^i, q_l^i) \geq (u_l^e, q_l^e)$. This proves part (i).

Now we show part (ii). If $u_h^I > (1-\theta_l)\underline{q}$, then the incumbent must offer two contracts. So we only need to consider the case $u_h^I \leq (1-\theta_l)\underline{q}$. Since $u_h^I > u_h^E$, we must have $u_h^E < (1-\theta_l)\underline{q}$. Now suppose we have an equilibrium in which the incumbent offers only an h contract and the entrant offers two contracts characterized by u_h^i, $(u_h^e, \overline{q})$ and (u_l^e, q_l^e), where $u_l^e > 0$. Note that for the entrant the DIC must be binding. Moreover, $u_h^e = u_l^e + (1-\theta_l)q_l^e > u_h^i$. This is because if $u_h^i \geq u_l^e + (1-\theta_l)q_l^e$ then the incumbent can profitably introduce a low contract. Let $\Delta u_h \equiv u_h^i - u_h^e < 0$.

For the above contracts to be an equilibrium, the incumbent should have no incentive to raise u_h^i to u_h^e and introduce a low contract (u_l^e, q_l^e). That is,

$$\alpha\left(\frac{1}{4} + \frac{\Delta u_h}{2k}\right)\left(\frac{1}{2} - u_h^i\right) \geq \alpha\frac{1}{4}\left(\frac{1}{2} - u_h^e\right) + (1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^e\right]. \tag{17}$$

Similarly, the entrant should have no incentive to reduce u_h^e to u_h^i and only offer h contract. That is,

$$\alpha\left(\frac{1}{4} - \frac{\Delta u_h}{2k}\right)\left(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^e\right) + (1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^e\right] \geq \alpha\frac{1}{4}\left(\overline{q} - \frac{1}{2}(\overline{q})^2 - u_h^i\right). \tag{18}$$

Rearrange the inequalities (17) and (18), we have

$$\alpha\left[-\frac{1}{4}\Delta u_h + \frac{\Delta u_h}{2k}\left(\frac{1}{2} - u_h^i\right)\right] \geq (1-\alpha)\frac{u_l^e}{k}\left[\theta_l q_l^1 - \frac{1}{2}(q_l^1)^2 - u_l^e\right] \geq \alpha\left[-\frac{1}{4}\Delta u_h + \frac{\Delta u_h}{2k}\left(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^e\right)\right]. \tag{19}$$

Given that $\Delta u_h < 0$, $\overline{q} - \frac{1}{2}(\overline{q})^2 < 1/2$, and both $(\frac{1}{2} - u_h^i) > 0$ and $(\overline{q} - \frac{1}{2}\overline{q}^2 - u_h^e) > 0$ (the profit margins for the high type are positive), we have that the first term of (19) is strictly less than the last term of (19), which contradicts the inequality of (19). This proves part (ii). ■

Proof of Proposition 6: From the full information solution when $k \leq \frac{2}{3}[1 - \frac{1}{2}(1-\overline{q})^2]$, we see that u_h^I is increasing in $\overline{q}$ and u_l^I is independent of $\overline{q}$. Thus if the unconstrained solution for the incumbent is feasible with $\overline{q}$, it must be feasible with $\overline{q}'$. Therefore, we only need to show that the results hold when the unconstrained solution is not feasible with both $\overline{q}$ and $\overline{q}'$. Note that in this case, the DIC must be binding with both $\overline{q}$ and $\overline{q}'$, and $k > \frac{2}{3}\theta_l^2$.

We first show that the incumbent must offer two contracts in the duopoly equilibrium with $\overline{q}'$. If $u_h^{I\prime} > (1-\theta_l)\underline{q}$, then the incumbent must offer two contracts with $\overline{q}'$. Thus we only need to consider the case that $u_h^{I\prime} \leq (1-\theta_l)\underline{q}$. Since $u_h^I < u_h^{I\prime}$, $u_h^I < (1-\theta_l)\underline{q}$. By part (ii) of Proposition 5, we know that in duopoly equilibrium it cannot be the case that the incumbent offers the h contract only and the entrant offers two contracts. Therefore, it is sufficient to rule out the case that with $\overline{q}'$ both firms offering h contract only cannot be an equilibrium.

First suppose that with $\overline{q}$ the entrant offers two contracts in the duopoly equilibrium. By Proposition 5, the incumbent will also offer two contracts. If $\overline{q}$ is increased to $\overline{q}'$, by inspecting entrant's FOC's (15) and (16) we can see that she will continue to offer both contracts, and thus the incumbent offers two contracts as well.

Now suppose initially the entrant offers one contract only. With $\overline{q}$ in the duopoly equilibrium the incumbent offers two contracts, hence given that the entrant offers h contract only with u_h^E, there is a $u_l^{i*} > 0$, $q_l^{i*} \geq \underline{q}$, and $u_h^{i*} = u_l^{i*} + (1-\theta_l)q_l^{i*} > u_h^I$ such that

$$\alpha\left(\frac{1}{4}+\frac{u_h^{i*}-u_h^E}{2k}\right)\left(\frac{1}{2}-u_h^{i*}\right)+(1-\alpha)\frac{u_l^{i*}}{k}\left[\theta_l q_l^{i*}-\frac{1}{2}(q_l^{i*})^2-u_l^{i*}\right] > \alpha\left(\frac{1}{4}+\frac{u_h^I-u_h^E}{2k}\right)\left(\frac{1}{2}-u_h^I\right). \tag{20}$$

The above inequality says that the incumbent has an incentive to offer a low contract instead of offering only an h contract. Let $\Delta u_h^I \equiv u_h^{I\prime} - u_h^I > 0$ and $\Delta u_h^E \equiv u_h^{E\prime} - u_h^E > 0$. By earlier results, $\Delta u_h^E = 2\Delta u_h^I$.

Now we show that with $\overline{q}'$, both firms offering only an h contract cannot be an equilibrium. It is sufficient to show that when the entrant offers h contract alone with $u_h^{E\prime}$, the incumbent's best response is to offer two contracts instead of offering h contract alone with $u_h^{I'}$. For this purpose, we construct the following two contracts for the incumbent: offering $u_h^{i*} + \Delta u_h^I$ to type h, and offering u_l^{i*} and q_l^{i*} to the low type. Note that these two contracts are not the best response among all the possibilities of offering two contracts (the DIC is not binding). Nevertheless, we show that these two contracts yield a higher profit to the incumbent than the best response of offering h contract alone.

That is,

$$\alpha\left(\frac{1}{4}+\frac{u_h^{i*}+\Delta u_h^I-u_h^{E\prime}}{2k}\right)\left(\frac{1}{2}-u_h^{i*}-\Delta u_h^I\right)+(1-\alpha)\frac{u_l^{i*}}{k}\left[\theta_l q_l^{i*}-\frac{1}{2}(q_l^{i*})^2-u_l^{i*}\right] \tag{21}$$
$$> \alpha\left(\frac{1}{4}+\frac{u_h^{I\prime}-u_h^{E\prime}}{2k}\right)\left(\frac{1}{2}-u_h^{I\prime}\right).$$

To see that (21) holds, it suffices to show that, for (20) and (21), the difference of the first terms, ΔA, equals the difference of the third terms, ΔB. Specifically,

$$\begin{aligned}\Delta A &= \frac{1}{4}\Delta u_h^I+\frac{\Delta u_h^E-\Delta u_h^I}{2k}\left(\frac{1}{2}-u_h^{i*}\right)+\frac{u_h^{i*}+\Delta u_h^I-u_h^{E\prime}}{2k}\Delta u_h^I,\\ \Delta B &= \frac{1}{4}\Delta u_h^I+\frac{\Delta u_h^E-\Delta u_h^I}{2k}\left(\frac{1}{2}-u_h^I\right)+\frac{u_h^{I\prime}-u_h^{E\prime}}{2k}\Delta u_h^I,\\ \Delta B-\Delta A &= \frac{\Delta u_h^E-2\Delta u_h^I}{2k}(u_h^{i*}-u_h^I)=0,\end{aligned}$$

where the last equality follows since $\Delta u_h^E = 2\Delta u_h^I$. Therefore, with $\overline{q}'$ both firms offering the h contract only cannot be an equilibrium; the incumbent must offer two contracts.

We next show that $q_l^{i\prime} \geq q_l^i$. Following the previous analysis, for the case that we are interested in, the programming problem is the same as (P), and the FOCs for the incumbent are given by (13)-(14). Let (u_l^i, q_l^i) and $(u_l^{i\prime}, q_l^{i\prime})$ be the solutions to (13)-(14) with $\overline{q}$ and $\overline{q}'$, respectively. With $\overline{q}$ whether the entrant offers h contract only or offers two contracts, when $\overline{q}$ increases to $\overline{q}'$, the entrant must respond optimally to (u_l^i, q_l^i) in a way that $u_h^{e\prime} > u_h^e$ because $\overline{q}' - \frac{1}{2}(\overline{q}')^2 > \overline{q} - \frac{1}{2}(\overline{q})^2$. Since with both $\overline{q}$ and $\overline{q}'$ the incumbent offers two contracts, we have $\mu^i = \mu^{i\prime} = 0$. With $u_h^{e\prime} > u_h^e$, from (13)-(14) we see that if $(u_l^{e\prime}, q_l^{e\prime}) = (u_l^e, q_l^e)$, the LHS of (13) and (14) (excluding $\mu^{i\prime}$ and $\lambda^{i\prime}$) are both strictly higher under $\overline{q}'$ than under $\overline{q}$. This implies that $(u_l^{i\prime}, q_l^{i\prime}) \geq (u_l^i, q_l^i)$ and $u_h^{i\prime} > u_h^i$. ■

Proof of Lemma 5: When $k \leq 1/2$, type h is fully covered. When $k \leq \frac{\theta_l^2}{2}$, the DIC's must bind; When $\frac{\theta_l^2}{2} < k \leq \frac{\theta_m^2}{2}$, the DIC$_{hm}$ must bind under full information, and the DIC$_{ml}$ binds if $\frac{k}{2} < \frac{\theta_l^2}{4}+(\theta_m-\theta_l)\theta_l$, which holds if $\theta_m < 3\theta_l$.[14] When $k > \frac{\theta_m^2}{2}$, the DIC$_{hm}$ binds if $\frac{k}{2} < \frac{\theta_m^2}{4}+(1-\theta_m)\theta_m$. Similarly, the DIC$_{ml}$ binds if $\frac{\theta_m^2}{4} < \frac{\theta_l^2}{4} + (\theta_m - \theta_l)\theta_l$, which again holds if $\theta_m < 3\theta_l$.[15]

Therefore, a set of sufficient conditions for both DIC's to bind is that $\theta_m < 3\theta_l$ and $\frac{k}{2} < \frac{\theta_m^2}{4} + (1-\theta_m)\theta_m$. We hence maintain these two assumptions in this subsection.

[14] To offer separate contracts, u_m must increase which makes DIC$_{hm}$ even more binding.

[15] When DIC$_{hm}$ binds, to offer separate contracts, u_m needs to be reduced which makes DIC$_{ml}$ more binding.

We first look at the case of fully separating equilibria. The programming problem is as follows:

$$\max_{(u_l,q_l,q_m)} \frac{\alpha_h}{2}\left[\frac{1}{2} - u_l - (\theta_m - \theta_l)q_l - (1-\theta_m)q_m\right] + \alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k}\left(\theta_m q_m - \frac{1}{2}q_m^2 - u_m\right)$$
$$+ \alpha_l \frac{u_l}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right)$$
subject to: $u_l \geq 0$; $q_m \geq \underline{q}$; $q_l \geq \underline{q}$

Let the Lagrangian multipliers of the three constraints be μ, λ_m, and λ_l respectively. The FOCs are as follows:

$$\begin{aligned}
-\frac{\alpha_h}{2} + \frac{\alpha_m}{k}\left(\theta_m q_m - \frac{1}{2}q_m^2 - 2u_m\right) + \frac{\alpha_l}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2 - 2u_l\right) + \mu &= 0, \\
\mu \geq 0,\ \mu = 0 \text{ if } u_l &> 0; \qquad (22) \\
-\frac{\alpha_h}{2}(1-\theta_m) + \frac{\alpha_m}{k}u_m(\theta_m - q_m) + \lambda_m &= 0, \\
\lambda_m \geq 0,\ \lambda_m = 0 \text{ if } q_l &> \underline{q}. \qquad (23) \\
-\frac{\alpha_h}{2}(1-\theta_l) + \frac{\alpha_m}{k}\left(\theta_m q_m - \frac{1}{2}q_m^2 - 2u_m\right)(1-\theta_l) + \frac{\alpha_l}{k}u_l(\theta_l - q_l) + \lambda_l &= 0, \\
\lambda_l \geq 0,\ \lambda_l = 0 \text{ if } q_l &> \underline{q}. \qquad (24)
\end{aligned}$$

From (23), we can see that if $\alpha_h(1-\theta_m) \geq \alpha_m(\theta_m - \underline{q})$, then $\lambda_m > 0$ and $q_m = \underline{q}$ (since $u_m/k \leq 1/2$). Therefore, $\alpha_h(1-\theta_m) \geq \alpha_m(\theta_m - \underline{q})$ implies that fully separating is not optimal. Moreover, if h and m contracts are offered only, $q_m = \underline{q}$.

Now consider the case of partial pooling (types m and l pool at the low contract). The programming problem is as follows:

$$\max_{(u_l,q_l)} \frac{\alpha_h}{2}\left[\frac{1}{2} - u_l - (1-\theta_l)q_l\right] + \left[\alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k} + \alpha_l \frac{u_l}{k}\right]\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right)$$
subject to: $u_l \geq 0$; $q_l \geq \underline{q}$

The FOCs are as follows:

$$-\frac{\alpha_h}{2}+\frac{\alpha_m}{k}\left[\theta_l q_l-\frac{1}{2}q_l^2-u_l-u_m\right]+\frac{\alpha_l}{k}\left[\theta_l q_l-\frac{1}{2}q_l^2-2u_l\right]+\mu = 0,$$

$$\mu\geq 0,\ \mu=0 \text{ if } u_l > 0; \qquad (25)$$

$$-\frac{\alpha_h}{2}(1-\theta_l)+\frac{\alpha_m}{k}(\theta_m-\theta_l)\left[\theta_l q_l-\frac{1}{2}q_l^2-u_l\right]+\left[\frac{\alpha_m}{k}u_m+\frac{\alpha_l}{k}u_l\right](\theta_l-q_l)+\lambda = 0,$$

$$\lambda\geq 0,\ \lambda=0 \text{ if } q_l > \underline{q}. \qquad (26)$$

From (26), we can see that if $\alpha_h(1-\theta_l)\geq\frac{\alpha_m}{k}(\theta_m-\theta_l)\theta_l^2+(1-\alpha_h)(\theta_l-\underline{q})$, then $\lambda>0$ and $q_l=\underline{q}$.

To establish that partial pooling is optimal, we must show that partial pooling dominates exclusion, that is, offering an h contract only or only offering h and m contracts. Offering an h contract alone leads to a (half) profit of $\pi_h=\frac{\alpha_h}{4}(1-k)$. When offering h and m contracts only, the optimal $u_m=\frac{1}{2}(\theta_m\underline{q}-\frac{1}{2}\underline{q}^2-\frac{\alpha_h}{\alpha_m}\frac{k}{2})$. Let the corresponding profit be π_{hm}. If

$$\theta_m\underline{q}-\frac{1}{2}\underline{q}^2\leq\frac{\alpha_h}{\alpha_m}\frac{k}{2}, \qquad (27)$$

then $u_m\leq 0$, which means that $\pi_{hm}<\pi_h$. In the case of partial pooling, the optimal u_l is given by

$$u_l=\frac{1}{2}\left[\theta_l\underline{q}-\frac{1}{2}\underline{q}^2-\frac{\alpha_h}{1-\alpha_h}\frac{k}{2}-\frac{\alpha_m}{1-\alpha_h}(\theta_m-\theta_l)\underline{q}\right].$$

Let the corresponding total profit be $\pi_{h(ml)}$. Given u_l, we have

$$\pi_{h(ml)}-\pi_h>-\frac{\alpha_h}{2}\left[u_l+(1-\theta_l)\underline{q}-\frac{k}{2}\right]+(1-\alpha_h)\frac{u_l}{k}\left(\theta_l\underline{q}-\frac{1}{2}\underline{q}^2-u_l\right)\equiv f(u_l).$$

The maximum $f(u_l)$, $f(u_l^*)$, can be calculated readily. Now if

$$f(u_l^*)=(1-\alpha_h)\frac{(\theta_l\underline{q}-\frac{1}{2}\underline{q}^2)^2-(\frac{\alpha_h}{1-\alpha_h}\frac{k}{2})^2}{4k}-\frac{\alpha_h}{2}\left[\frac{1}{2}\left(\theta_l\underline{q}-\frac{1}{2}\underline{q}^2-\frac{\alpha_h}{1-\alpha_h}\frac{k}{2}\right)+(1-\theta_l)\underline{q}-\frac{k}{2}\right]>0, \qquad (28)$$

then $\pi_{h(ml)}>\pi_h$. Overall, if both (27) and (28) hold, then we have $\pi_{h(ml)}>\pi_h>\pi_{hm}$. That is, partial pooling is optimal. ■

Proof of Proposition 8 : Now suppose both firms offer three separate contracts. The problem becomes:

$$\begin{aligned}\max \quad & \alpha_h \left[\tfrac{1}{4} + \tfrac{u_l+(\theta_m-\theta_l)q_l+(1-\theta_m)q_m-u_h^{\mathrm{II}}}{2k}\right]\left(\tfrac{1}{2} - u_l - (\theta_m-\theta_l)q_l - (1-\theta_m)q_m\right) \\ & +\alpha_m \tfrac{u_l+(\theta_m-\theta_l)q_l}{k}\left[\theta_m q_m - \tfrac{1}{2}q_m^2 - u_l - (\theta_m-\theta_l)q_l\right] + \alpha_l \tfrac{u_l}{k}\left[\theta_l q_l - \tfrac{1}{2}q_l^2 - u_l\right] \\ \text{s.t.} \quad & u_l \geq 0,\ q_m \geq \underline{q},\ q_l \geq \underline{q}\end{aligned}$$

The LHS (excluding λ_m) of the FOC for q_m is:

$$(u_l + (\theta_m - \theta_l)q_l)\left[\frac{\alpha_m}{k}(\theta_m - q_m) - \alpha_h \frac{1-\theta_m}{2k}\right] + \alpha_h \frac{1-\theta_m}{2k}\left[\frac{1-k}{2} - (1-\theta_m)q_m\right].$$

If

$$\frac{\alpha_m}{k}(\theta_m - \underline{q}) - \alpha_h \frac{1-\theta_m}{2k} < 0, \tag{29}$$

and

$$\frac{1-k}{2} - (1-\theta_m)\underline{q} < 0, \tag{30}$$

then $q_m = \underline{q}$, which means that fully separating is not optimal.

Next consider the case where partial pooling occurs. The problem now becomes:

$$\begin{aligned}\max \quad & \alpha_h \left[\frac{1}{4} + \frac{u_l + (1-\theta_l)q_l - u_h^{\mathrm{II}}}{2k}\right]\left(\frac{1}{2} - u_l - (1-\theta_l)q_l\right) \\ & + \left[\alpha_m \frac{u_l + (\theta_m - \theta_l)q_l}{k} + \alpha_l \frac{u_l}{k}\right]\left(\theta_l q_l - \frac{1}{2}q_l^2 - u_l\right) \\ \text{s.t.} \quad & u_l \geq 0, q_m \geq \underline{q}, q_l \geq q\end{aligned}$$

The FOC for u_l is as follows:

$$\frac{\alpha_h}{2k}\left[\frac{1-k}{2} - (1-\theta_l)q_l\right] + \frac{1-\alpha_h}{k}\left(\theta_l q_l - \frac{1}{2}q_l^2\right) - \frac{\alpha_m}{k}(\theta_m - \theta_l)q_l - \left[\frac{\alpha_h}{2k} + 2\frac{1-\alpha_h}{k}\right]u_l + \mu = 0$$

And the LHS (excluding λ) of the FOC for q_l is:

$$\alpha_h \frac{1-\theta_l}{2k}\left[\frac{1-k}{2} - (1-\theta_l)q_l\right] + \alpha_m \frac{\theta_m - \theta_l}{k}\left(2\theta_l q_l - \frac{3}{2}q_l^2\right) + \left[\frac{1-\alpha_h}{k}(\theta_l - q_l) - \alpha_h \frac{1-\theta_l}{2k} - \alpha_m \frac{\theta_m - \theta_l}{k}\right]u_l$$

If

$$(1-\alpha_h)(\theta_l - \underline{q}) - \frac{\alpha_h}{2}(1-\theta_l) - \alpha_m(\theta_m - \theta_l) < 0,$$

which is equivalent to:

$$(\theta_l - \underline{q}) - \alpha_h\left(\frac{\theta_l + 1}{2} - \underline{q}\right) < \alpha_m(\theta_m - \theta_l), \tag{31}$$

then the LHS of the FOC for q_l is less than or equal to

$$\alpha_h \frac{1-\theta_l}{2k}\left[\frac{1-k}{2} - (1-\theta_l)q_l\right] + \alpha_m \frac{\theta_m - \theta_l}{k}\left(2\theta_l q_l - \frac{3}{2}q_l^2\right).$$

Define $A = \alpha_h \frac{1-\theta_l}{2}$ and $B = \alpha_m(\theta_m - \theta_l)$. Then the above expression is proportional to

$$-\frac{3B}{2}q_l^2 + (2\theta_l B - (1-\theta_l)A)q_l + \frac{1-k}{2}A.$$

The above expression is decreasing in q if $2\theta_l B - (1-\theta_l)A < 0$, or more explicitly,

$$4\theta_l(\theta_m - \theta_l)\alpha_m < (1-\theta_l)^2\alpha_h. \tag{32}$$

Therefore, if (32) and the following condition hold,

$$\alpha_h \frac{1-\theta_l}{2}\left[\frac{1-k}{2} - (1-\theta_l)\underline{q}\right] + \alpha_m(\theta_m - \theta_l)\left(2\theta_l\underline{q} - \frac{3}{2}\underline{q}^2\right) < 0, \tag{33}$$

then we have $q_l = \underline{q}$.

Next we will compare the expected profit from partial pooling with those from offering high contract only and offering both high and middle contracts.

If only the high contract is offered, the expected profit would be $\pi_h = \frac{\alpha_h k}{8}$. If both high and middle contracts are offered, the LHS (excluding multiplier) of the FOC for q_m is

$$u_m\left[\frac{\alpha_m}{k}(\theta_m - q_m) - \alpha_h\frac{1-\theta_m}{2k}\right] + \alpha_h\frac{1-\theta_m}{2k}\left[\frac{1-k}{2} - (1-\theta_m)q_m\right].$$

From condition (29) and (30), we have $q_m = \underline{q}$.

Given $q_m = \underline{q}$, the LHS (excluding multiplier) of the FOC for u_m is

$$\alpha_h\frac{1}{2k}\left[\frac{1-k}{2} - u_m - (1-\theta_m)\underline{q}\right] + \frac{\alpha_m}{k}\left[\theta_m\underline{q} - \frac{1}{2}\underline{q}^2 - 2u_m\right].$$

If

$$\frac{\alpha_h}{2}\left(\frac{1-k}{2} - (1-\theta_m)\underline{q}\right) + \alpha_m\left(\theta_m\underline{q} - \frac{1}{2}\underline{q}^2\right) < 0, \tag{34}$$

then the optimal $u_m = 0$ which means that it is not profitable to offer a middle contract along with high contract.

With partial pooling, we denote expected profit as $\pi_{h(ml)}$. From the previous discussion, we know that

$$\begin{aligned}\pi_{h(ml)} - \pi_h &= \frac{\alpha_h}{4}\left[\frac{1-k}{2} - u_l - (1-\theta_l)\underline{q}\right] + \left[(1-\alpha_h)\frac{u_l}{k} + \frac{\alpha_m}{k}(\theta_m - \theta_l)\underline{q}\right]\left(\theta_l\underline{q} - \frac{\underline{q}^2}{2} - u_l\right) \\ &\equiv g(u_l);\end{aligned}$$

When

$$\max g(u_l) = \frac{\left[(1-\alpha_h)(\theta_l \underline{q} - \frac{\underline{q}^2}{2}) - \frac{\alpha_h}{4}k - \alpha_m(\theta_m - \theta_l)\underline{q}\right]^2}{4(1-\alpha_h)} + \frac{\alpha_h}{4}\left(\frac{1-k}{2} - (1-\theta_l)\underline{q}\right) + \frac{\alpha_m}{k}(\theta_m - \theta_l)\underline{q}(\theta_l \underline{q} - \frac{\underline{q}^2}{2}) \geq 0, \quad (35)$$

partial pooling is optimal in duopoly.

We can simplify the conditions a little bit. First, condition (30) implies that $(\theta_m - \frac{2}{3})^2 < \frac{6k-2}{9}$, which is one sufficient condition for binding DICs. Second, condition (29), (32), (33), and (34) are all about the proportions of high type and middle type, and they can be summarized by the following condition:

$$\alpha_h > \delta \alpha_m, \quad (36)$$

where

$$\delta = \max\left\{\frac{2(\theta_m - \underline{q})}{1-\theta_m}, \frac{4\theta_l(\theta_m - \theta_l)}{(1-\theta_l)^2}, \frac{2(\theta_m - \theta_l)(2\theta_l \underline{q} - \frac{3}{2}\underline{q}^2)}{(1-\theta_l)((1-\theta_l)\underline{q} - \frac{1-k}{2})}, \frac{2(\theta_m \underline{q} - \frac{1}{2}\underline{q}^2)}{(1-\theta_m)\underline{q} - \frac{1-k}{2}}\right\}.$$

■

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