Contributions to Theoretical Economics

Optimal Auctions with Endogenous Entry

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Abstract

We consider a single object, independent private value auction model with entry. Potential bidders are ex ante symmetric and randomize about entry. After entry, each bidder incurs a cost, then learns her private value and a set of signals that may lead to updated beliefs about other entrants' valuations. It is shown that the Vickrey auction with free entry maximizes the expected revenue. Furthermore, if the information potentially available to bidders after entry is sufficiently rich, then the Vickrey auction, up to its equivalent class, is also the only optimal sealed-bid auction.

KEYWORDS: Auctions, Vickrey auction, robust mechanism design, efficient entry, complete classes

1 Introduction

We consider an auction model in which the object is a single indivisible asset with symmetric independent private value (IPV). There is an entry cost for each potential bidder, which can be interpreted as the cost of preparing a bid or estimating the valuation of the asset, etc. There are a large number of potential bidders, who randomize about entry so that the ex ante expected profit from attending the auction is zero. Before entry, bidders know nothing except a common prior (a distribution) about the valuation of the asset. After entry, each bidder incurs an entry cost and then learns her private value (t_i) and in addition, a set of signals (s) that may lead to updated beliefs (distributions) about entrant bidders' valuations.[1] In such a setup, we show that the Vickrey auction (the second-price sealed bid auction) with free entry maximizes the expected revenue. Furthermore, the Vickrey auction, up to its equivalent class, is also the only optimal sealed-bid auction if the information potentially available to bidders after entry is sufficiently rich, in the sense that the signal system can generate a "complete class" of posterior distributions.

Our model extends the existing literature on auctions with costly entry. In earlier work (e.g., Johnson (1979), French and McCormick (1984), McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), Levin and Smith (1994)) entry is usually formulated such that potential bidders do not possess private information, and after incurring an entry cost, each entrant bidder learns a private signal about the value of the object. Our model differs from these formulations in one important respect: while beliefs (distributions about other bidders' valuations) are usually assumed to be fixed both before and after entry occurs, in our model bidders can update their beliefs after entry. We believe that this setup is not only theoretically more general, but also practically more realistic. For example, in some procurement auctions in the construction industry, contestants are usually not certain about their competitors' identities until they get into the final stage, which is a costly process to prepare blueprints or detailed proposals. During this "due diligence" stage, they can in general learn more about their rivals' positions such as their capacities, overhead costs, and even reputations regarding whether they are "tough" or "soft" in the final bid tender, etc. This sort of "belief" updating can greatly enhance the

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[1]Our setup is thus different from the traditional literature on optimal auctions and revenue comparisons which generally assumes that there is a fixed set of bidders, and that the bidders are endowed with information about their valuations (see, for example, Vickrey (1961), Riley and Samuelson (1981), Myerson (1981), Milgrom and Weber (1982)).

contestants' information before the final auction is conducted.[2]

More specifically, in our analysis we focus on symmetric entry equilibria, in which each potential bidder randomizes about entry with the same probability. The symmetric entry equilibrium was first introduced by Milgrom (1981) and received thorough analysis in Levin and Smith (1994). In any symmetric entry equilibrium, the expected rent to the bidders is driven down to zero by endogenous entry; hence the expected revenue to the seller is the same as the expected total surplus generated from the sale. This implies that any optimal auction must be efficient, which tremendously simplifies our job in searching for optimal auctions. We first show that any optimal auction must be ex post efficient. We then show that any efficient, incentive compatible (IC) direct revelation mechanism (DRM) is equivalent to a Vickrey-Clarke-Groves (VCG) mechanism from the interim perspective (i.e., at the point in time when each bidder has just learned her private information). Finally, we show that if the (belief) signal system generates a "complete class" for the payment rule space, then the above equivalence also holds in an "almost everywhere" sense. In other words, any optimal mechanism in this environment must be essentially a VCG mechanism, which in turn implies that the optimal auction in our setting can only be the Vickrey auction, up to its equivalent class.

Our result thus provides a new explanation for the prevalence of simple auctions (English auctions and Vickrey auctions).[3] It is well known that simple auctions are not optimal in the usual IPV settings (Myerson, 1981). However, by restricting to certain (narrower) classes of auction mechanisms, the optimality of simple auctions can be established. For example, it has been shown that English auctions are optimal among simple sequential auctions (Lopomo, 1998), and within the class of all posterior-implementable trading mechanisms (Lopomo, 2000). Even in more general settings where simple auctions are known to be not revenue-maximizing, Neeman (2003) shows that they can be quite "close" to full optimality.[4] In our model, by taking endogenous entry into account, we show that the simple auctions themselves are optimal, and are

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[2]Bid tabs provide company records of who bid what on construction contracts. The bid tabs underlying the data analysis in Dyer and Kagel (1996) demonstrate the difficulty of competing against a particular low overhead rival. Firms would know, from both their sub-contractors and when they picked up the blueprints for a job, who else was bidding on the job. In particular, if this low overhead rival was also bidding on the same job.

[3]Note that in the IPV setting, an English auction is outcome-equivalent, though not strategically equivalent, to a Vickrey auction.

[4]Specifically, Neeman (2003) defines a concept of effectiveness as a measure of the proximity to optimality, and shows that the effectiveness of the simple auctions can be quite high for a wide range of distributions of valuations.

uniquely optimal up to its equivalent class. This provides another explanation for the prevalence of simple auctions.

Since our search for optimal auctions is reduced to the search within the class of ex post efficient mechanisms, this relates our work to the recent literature on robust mechanism design which focuses on ex post implementation or dominant strategy implementation (see, for example, Dasgupta and Maskin (2000), Perry and Reny (2002), Segal (2003) in auction settings, and in particular, Bergemann and Morris (2004) in more general mechanism design settings).[5] Our contribution to this literature is twofold. First, we suggest an environment in which robust mechanism is not only relevant for efficiency, but also important for revenue maximization. Our story is fairly simple: in an auction environment with endogenous entry, the optimal auction is one that maximizes ex ante expected total surplus, which is in turn one that is ex post efficient. This also relates our work to Chung and Ely (2004), who provide alternative rationales for employing dominant strategy mechanisms in auction settings when the auctioneer's goal is revenue maximization.[6] Second, the notion of complete classes developed in this research can be used to identify conditions for the equivalence between Bayesian implementation and dominant strategy implementation. The complete classes in our setting can be interpreted as some tighter conditions compared to the conditions of all common-prior type spaces suggested by Bergemann and Morris (2004).[7]

The paper is organized as follows. Section 2 presents the model. Section 3 shows the optimality of the Vickrey auction with free entry. Section 4 identifies conditions under which the Vickrey auction, up to its equivalent class, is also the unique optimal auction. Section 5 is a discussion and Section 6 concludes.

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[5]Ex post implementation is equivalent to dominant strategy implementation in private value settings. Bergemann and Morris (2004) consider more general settings with interdependent values. In private value environments, the equivalence between Bayesian implementation and dominant strategy implementation for all-prior type space was observed in Ledyard (1978,1979), Dasgupta, Hammond and Maskin (1979), and Groves and Ledyard (1987).

[6]Roughly speaking, their idea is that if the auctioneer does not have reliable information about the bidders' beliefs, then under certain conditions, dominant strategy mechanisms can be used against the worse scenario (maxmin foundation), or they can be justified as the optimal choices with respect to some subjective beliefs consistent with Bayesian rationality (Bayesian fundation).

[7]Bergemann and Morris allow for a full range of higher order beliefs, while in our analysis it is enough to have a range of second order beliefs.

2 The Model

We consider a mechanism through which a single indivisible asset is offered for sale to N potentially interested buyers. Ex ante, the potential buyers are symmetric and hold a common prior that their private values will be independent draws from a distribution $G(\cdot)$. In contrast to the traditional auction model, we augment the selling mechanism with an entry game in which information acquisition is costly. Suppose that n potential buyers decide to attend the auction. Then after incurring an entry cost c, each entrant bidder learns her private value t_i, and a signal vector $s = (s_1, \cdots, s_n)$, which, combined with the initial belief $G(\cdot)$, will lead to updated beliefs about entrant bidders' valuations. For example, suppose ex ante that each potential buyer holds a belief that the valuation will be drawn from Uniform$[0, 1]$; after entry, bidder i's belief is updated such that entrant j's value is drawn from Uniform$[0, s_j]$ for all $j \in E \setminus \{i\}$, where $E = \{1, \cdots, n\}$ is the index set of entrant bidders.[8]

s is a set of public signals known to the entrant bidders. Since each bidder knows her own valuation (t_i), it is not essential to assume that she also learns s_i. However, for ease of equilibrium analysis, we assume that each bidder (i) knows her own signal s_i as well. In this way each bidder also knows what other bidders think about her (in terms of beliefs about her valuation). We assume that the conditional distribution of S_i given the list of other variables $(t_1, \cdots, t_n, s_{-i})$ depends only on t_i. (A special case is the standard model in which $S_i = t_i$; that is, the signals precisely identify the underlying types.) Formally, we assume that conditional on t_i, S_i is drawn from distribution $F(\cdot|t_i)$. With abuse of notation, we denote the support of S_i given t_i as $Supp(F(\cdot|t_i))$, and the unconditional support of S_i as $\mathcal{S}_i$.

For our main results to hold, we do not need to specify the exact structure of belief updating, though Bayesian updating would be a natural way for the formulation. We require only that the signal generating system (denoted as $\{\mathcal{S}\}$) be consistent, in the sense that it gives rise to the ex ante common prior $G(\cdot)$.

The seller's own valuation is normalized to be zero. Bidder i's valuation $t_i \in [0, \bar{t}_i](=: T_i)$, $i \in E$. Both the seller and the buyers are assumed to be risk-neutral. In this paper, we also assume that N or c is large enough so that if all potential buyers enter the auction with probability one, their expected

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[8]In a recent paper, Fang and Morris (2004) analyze two-bidder auction games with similar information structure, in which each bidder knows her own private value and a noisy signal about the other bidder's value. They characterize equilibria under both first-price and second-price auctions.

profits will be strictly negative. This assumption distinguishes our model from traditional auction models that do not consider entry cost.

An auction mechanism consists of a set B_i of bids (messages) for each bidder $i = 1, \cdots, n$, an allocation rule $p : B \rightarrow [0, 1]^n$, and a payment rule $\xi : B \rightarrow R^n$. Each bidder will select a bid $b_i \in B_i$, and given a bid profile $b = (b_1, .., b_n)$, $p_i(b) \in [0, 1]$ is bidder i's probability of winning the object, and $\xi_i(b)$ is the monetary payment that bidder i makes to the seller.

In this paper, we shall restrict our search for optimal auctions within the class of standard auctions, which is defined below:

Definition 1 *(Standard auctions:)* Standard auctions are sealed-bid auctions in which: (1) bidders are required to submit nonnegative real numbers as their bids; (2) the bidder with the highest bid wins the object; (3) payments depend deterministically on bids; (4) the payment to the bidder with the lowest possible type is zero.

(1) and (4) are common in the literature. (2) and (3) specify that both the allocation and payment rules are functions of bids only, hence the rules of our standard auctions, in particular, cannot be contingent on belief signal s. This requirement is consistent with the spirit of Wilson's Doctrine (Wilson, 1987), who suggest that the trading rules not depend on particular "environments" (e.g., common prior, bidders' probability assessments about each other's private information). Indeed, real world auction rules typically process bids only and rarely involve distributions. In our setting, since s is not realized when the selling mechanism is announced, it would be more reasonable to assume that feasible auctions are those not contingent on s.[9]

At the outset of the game the seller moves first by announcing the selling mechanism (which includes the auction format that is consistent with Def-

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[9]If we take into account some "costs" that are not modeled in this paper, restricting the search for optimal auctions to the standard auctions is without loss of generality. First, it would be costly for the seller to learn s directly; Second, though one may argue that the seller can make the bidders reveal s and hence learn s indirectly (for example, through forcing mechanism proposed by Cremer and McLean (1988), McAfee, McMillan, and Reny (1989), and McAfee and Reny (1992)), in practice there are some relevant costs that favor mechanisms that reveal as little information as possible. For example, there is some small probability of information leakage that would disadvantage a bidder in another setting, say a bargaining environment, in which the other bargainer is uninformed about the value distribution. Then, the seller would bear a cost of forcing information revelation by discouraging entry. In either case above, the auctioneer may find it sub-optimal to make a selling mechanism contingent on s given the optimality of the Vickrey auction established in Section 3.

inition 1, the reserve prices, the entry fees, and so on). Taking the selling mechanism as given, the potential buyers make entry decisions. Finally the entrant bidders submit sealed bids. The seller's objective is to maximize expected revenue.

Since potential buyers are ex ante symmetric, in the following analysis we will focus exclusively on *symmetric entry equilibrium*, in which each potential buyer enters the auction with the same probability q^*, where q^* is determined by N, c, $G(\cdot)$ and the selling mechanism.[10]

3   The Optimality of the Vickrey Auction

Since the selling mechanism must be announced before s is realized, an optimal auction in this context must maximize the expected revenue regardless of the post-entry environment (or the realization of S). In this section we establish that the Vickrey auction is one such optimal auction.

Theorem 1 *The Vickrey auction with free entry maximizes the expected revenue.*

Proof: Let $ER, E\Pi$, and ES be the expected revenue to the seller, the expected profit to the bidders, and the expected total surplus generated from the sale (net of entry cost), respectively. Then we have $ES = ER + E\Pi$. In any symmetric entry equilibrium, the randomization condition implies that $E\Pi = 0$. Therefore we have in equilibrium $ES = ER$, that is, the seller's expected revenue is the same as the expected surplus generated from the sale. Hence it suffices to show that the Vickrey auction with free entry generates the maximal possible expected surplus.

Let S_v be the surplus generated by a Vickrey auction, and S_a the surplus generated by an alternative auction. The Vickrey auction is ex post efficient.[11]

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[10]Note that despite the ex ante symmetry, the potential buyers' entry decision could be asymmetric. For example, there are equilibria where exactly n^* buyers enter the auction and $N - n^*$ buyers stay out (see, e.g., Johnson (1979), McAfee and McMillan (1987), and Engelbrecht-Wiggans (1993)). Smith and Levin (2002) use an experimental approach to test how people would coordinate with entry. Their results strongly reject the hypothesis of asymmetric entry (deterministic entry) and tend to favor the alternative hypothesis that entry is symmetric (stochastic entry).

[11]We assume that in equilibrium, each bidder employs the dominant strategy of bidding her true valuation (see Blume and Heidhues (2004) for other equilibria under a Vickrey auction).

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Thus conditional on n, the number of actual bidders, we have

$$S_v \geq S_a \tag{1}$$

Hence,

$$E(S_v|n) \geq E(S_a|n) \tag{2}$$

Note that when each potential buyer randomizes about entry with probability q (recall that we focus on symmetric entry), the actual number of entrants n is distributed as Binomial (N, q). Hence inequality (2) implies

$$E(S_v|q) \geq E(S_a|q) \quad \forall q \in (0,1) \tag{3}$$

Next we will show, following arguments paralleling those in Levin and Smith (1994), that q_v, the equilibrium entry probability under a Vickrey auction with free entry, maximizes $E(S_v|q)$.

Given the entry probability q, define p_n to be the probability that exactly n bidders enter the auction, then

$$p_n = \binom{N}{n} q^n (1-q)^{N-n}$$

We also let $t_{j;n}$ denote the jth highest order statistic among n *iid* valuation samples. Under a Vickrey auction with entry fee e, the entry condition (zero profit condition) is given by:

$$\sum_{n=1}^{N} p_n E(t_{1;n} - t_{2;n}) = Nq(c+e) \tag{4}$$

In a symmetric IPV setting one can show the following identity:[12]

$$E(t_{1;n} - t_{1;n-1}) = \frac{1}{n} E(t_{1;n} - t_{2;n}) \tag{5}$$

Substituting (5) into (4), and letting $e = 0$, we have

$$\sum_{n=1}^{N} p_n n(Et_{1;n} - Et_{1;n-1}) = Nqc \tag{6}$$

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[12]The intuition is that drawing one more value from a distribution and adding it to the previously drawn $n-1$ values will enable this newly drawn value to be the highest one with probability $1/n$.

On the other hand, the expected total surplus conditional on q is given by:

$$E(S_v|q) = \sum_{n=1}^{N} p_n(Et_{1;n} - nc) = \sum_{n=1}^{N} p_n Et_{1;n} - Nqc$$

Taking partial derivative with respect to q, and using (6), we have

$$
\begin{aligned}
\frac{\partial E(S_v|q)}{\partial q} &= \frac{1}{q(1-q)} \left[\sum_{n=1}^{N} p_n Et_{1;n}(n - qN) - qNc(1-q) \right] \\
&= \frac{1}{q(1-q)} \left[\sum_{n=1}^{N} p_n Et_{1;n}(n - qN) - (1-q) \sum_{n=1}^{N} p_n n(Et_{1;n} - Et_{1;n-1}) \right] \\
&= \frac{1}{q} \left[\sum_{n=1}^{N} p_n n Et_{1;n-1} - \sum_{n=1}^{N-1} p_{n+1}(n+1) Et_{1;n} \right] \\
&= 0
\end{aligned}
\tag{7}
$$

This shows that q_v, the equilibrium probability induced by free entry, maximizes the expected total surplus under a Vickrey auction. Combining this with (3) we have $E(S_v|q_v) \geq E(S_a|q)$ $\forall q \in (0,1)$. In particular, $E(S_v|q_v) \geq E(S_a|q_a)$, where q_a is the equilibrium probability of entry induced by the alternative selling mechanism.

Hence the Vickrey auction with free entry maximizes the expected total surplus. *Q.E.D.*

The Vickrey auction does not only achieve ex post efficient allocation, but also induces "first best" entry with zero entry fee (in the sense of achieving unconstrained optimal entry probability). Therefore it maximizes the expected surplus (and hence the expected revenue) in this costly entry environment.

To understand the optimality of free entry under the Vickrey auction, let's consider the case of deterministic entry and ignore the integer constraint of n for the moment. Given an entry fee e, bidders will enter the auction until the expected rent is driven down to zero. That is

$$\frac{1}{n} E(t_{1;n} - t_{2;n}) = c + e \tag{8}$$

Substituting Eq. (5) into (8), we have[13]

$$E(t_{1;n} - t_{1;n-1}) = c + e \tag{9}$$

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[13]Note that this substitution is not "exact," since identity (5) only makes sense when n is an integer. We are being vague here in order to get a rough intuition.

Therefore, when the entry fee $e = 0$, the marginal social gain of entry $E(t_{1;n} - t_{1;n-1})$ is exactly equalized by the marginal social cost of entry (c) in equilibrium. This implies that zero entry fee induces efficient entry in the deterministic entry case.

Now when symmetric entry is assumed, the randomization of each bidder's entry decision serves to "smooth" the integer problem and the above intuition still carries over. So the Vickrey auction with free entry is optimal.

The optimality of Vickrey auction with free entry is implied in Levin and Smith (1994). Theorem 1 extends this result to the setting where beliefs can also be updated throughout entry.

The proof of Theorem 1 also suggests the following Corollary, which will be most helpful for the analysis in the next section.

Corollary 1 *Any optimal auction must be (ex post) efficient.*

4 The Uniqueness of the Optimal Auction

By Corollary 1, the (ex post) efficiency is necessary for a mechanism to be optimal. Since beliefs can be updated after entry, the final auction is in general an asymmetric one. This would impose a strong restriction on the class of mechanisms that can induce efficient allocation. For example, under a first-price auction the equilibrium would be very sensitive to the asymmetry among bidders, and the outcome would often be inefficient.[14] This may narrow down the set of optimal auctions. On the other hand, as is well-known, a Vickrey auction always induces ex post efficient allocation in private value settings. This is true regardless of the asymmetry among bidders. (This is true even when bidders possess inconsistent beliefs about one another's valuations.) We thus ask the following question: Is the Vickrey auction also the only sealed-bid auction that achieves optimality in the class of standard auctions defined in the previous section?[15] It turns out that we can indeed identify exact conditions under which the Vickrey auction, up to its equivalent class, is the unique optimal sealed-bid auction.

The Revelation Principle implies that to identify the set of social choice functions that are implementable (in dominant strategies or Bayesian Nash

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[14]Usually "weak" bidders would bid more aggressively than "strong" bidders, as shown in Griesmer *et al.* (1967), Plum (1992), Lebrun (1999), and Maskin and Riley (2000).

[15]Beyond the class of sealed-bid auctions, we know that the Vickrey auction cannot be the unique optimal auction, since the English ascending bid auction is outcome-equivalent to a Vickrey auction.

equilibrium strategies), we need only identify those that are truthfully implementable. Without loss of generality, to identify optimal auctions we henceforth focus on the analysis of the incentive compatible (IC) direct revelation mechanism (DRM). By Corollary 1 and Definition 1, we can further restrict our attention to the IC efficient DRM that depends only on the report about t. Note that the seller chooses the auction format which is a standard auction defined in Definition 1, thus he may not know the induced DRM since he may not know s. However, the bidders observe s and hence can infer the induced DRM. For this reason we can still carry out DRM analysis.[16]

Given a report profile on types $t = (t_i, t_{-i})$, a DRM is described by an allocation rule $y(t) = \{y_i(t)\}_{i=1}^n$ and a payment rule $x(t) = \{x_i(t)\}_{i=1}^n$, where $y_i(t) \in [0,1]$ is bidder i's probability of winning the object, and $x_i(t)$ is the monetary payment that bidder i makes to the seller when t is reported. Write the social choice function $f(t) = (y(t), x(t))$.[17]

In any DRM characterized by allocation rule $y(\cdot)$ and payment rule $x(\cdot)$, if bidder i reports t_i' while all the other bidders report truthfully, then bidder i's payoff (or utility) is

$$U_i(t_i', t_{-i}; t_i) = u_i(f(t_i', t_{-i}); t_i) = t_i \cdot y_i(t_i', t_{-i}) - x_i(t_i', t_{-i}) \qquad (10)$$

Define

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\begin{aligned}
q_i(t_i'|s_{-i}) &= E_{t_{-i}}(y_i(t_i', t_{-i})|S_{-i} = s_{-i}) \text{ and} \\
m_i(t_i'|s_{-i}) &= E_{t_{-i}}(x_i(t_i', t_{-i})|S_{-i} = s_{-i})
\end{aligned}
\qquad (11)
$$

to be, respectively, bidder i's interim expected probability of winning and interim expected payment by reporting t_i' conditional on (t_i, s_{-i}) and everyone else reporting t_{-i} truthfully.

By standard arguments based on the Envelope Theorem, incentive compatibility implies[18]

$$m_i(t_i|s_{-i}) = t_i \cdot q_i(t_i|s_{-i}) - \int_0^{t_i} q_i(z|s_{-i}) \, dz \qquad (12)$$

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[16]The rules of standard auctions are not contingent on the realization of s, but the induced mechanisms may depend on s. For example, the rules of a first-price auction are not contingent on s, but the induced IC mechanism depends on s if s_i's are the same (in which case the post-entry environment is symmetric).

[17]Again, we are considering DRMs that are induced by a standard auction, hence both $y(t)$ and $x(t)$ may depend on s. To economize on notation, we do not explicitly recognize the dependence on s here.

[18]A generalized version of the Envelope Theorem is given by Milgrom and Segal (2002).

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As can be seen from (12), the interim expected payment rule in any incentive compatible DRM is completely determined by the interim expected allocation rule $q(\cdot)$, which is in turn determined by the allocation rule $x(\cdot)$.

Since any two efficient allocation rules coincide almost everywhere, Eq. (12) implies that any IC efficient DRM must be equivalent to a Vickrey-Clarke-Groves (VCG) mechanism at the point in time when each entrant has just learned (t_i, s_{-i}). We thus have the following lemma:

Lemma 1 *Any optimal DRM is payoff-equivalent to a VCG mechanism at the point in time when each entrant has just learned (t_i, s_{-i}).*

Without the restriction that the payment to the lowest possible type is zero, the above result holds more generally: from the interim perspective any IC efficient DRM is payoff-equivalent to a Groves' mechanism where the payment to the lowest type may vary arbitrarily (Mookherjee and Reichelstein, 1992. Also see Williams, 1999 for the equivalence under more general utility functions).

Lemma 1 implies that under interim expectation, the payment rule of any optimal DRM must be equivalent to the payment rule in a VCG mechanism. We next show that if the signal system is so "rich" that it generates a "complete class," then the above equivalence result also holds (almost everywhere) after removing the expectations. We formally introduce the definition of complete classes for a functional space $\mathcal{F}$ (defined on T) below:

Definition 2 *($\mathcal{F}$-Complete classes:)* A functional family $\mathcal{H}$ (defined on T) is a complete class for a functional space $\mathcal{F}$ if for $f, g \in \mathcal{F}$, that $\int fh = \int gh$ for all $h \in \mathcal{H}$ implies $f = g$ a.e.

As an example, by standard dual space arguments, L^q is a complete class for L^p (or L^q is a L^p-complete class), where $1 \leq p < \infty$, and $1/p + 1/q = 1$. In our problem, $\mathcal{F}$ is the payment rule functional space, and $\mathcal{H}$ is a conditional distribution family that is generated from the post-entry signal system (denoted as $\{\mathcal{S}\}$). More specifically, let $h(t|s)$ be t's density function conditional on the observed s, and let $\{H(t|\mathcal{S})\}$ be the family containing all the conditional densities arising from any possible signal $s \in \mathcal{S}$. Then a (conditional) distribution family $\{H(t|\mathcal{S})\}$ forms a complete class for the payment rule space $\mathcal{F}$ (defined on T) if for $f, g \in \mathcal{F}$, that $E(f(t)|s) = E(g(t)|s)$ for all $s \in \mathcal{S}$ implies $f(t) = g(t)$ a.e.

Since t is bounded, the feasible payment rules are also bounded. More generally, in this paper we will consider the payment rules that are *essentially*

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bounded. Formally, let $x(t)$ and $x^{'}(t)$ be two payment rules under two given mechanisms, then $x, x^{'} \in L^\infty(T)$, where $L^\infty(T)$ is the space of all measurable functions defined on T that are bounded except possibly on a subset of measure zero.

Let $\{\mathcal{S}_j\}$ denote the signal system that can generate any possible signal about $t_j \in T_j, j \in E$. The following assumption about the post-entry signal system is central for the proof of the uniqueness theorem:

Assumption (CC): *For each entrant $i \in E$, the signal system $\{\mathcal{S}_{-i}\}$ generates a complete class for $L^\infty(T_{-i})$.*

Two complete classes in this context are stated as follows:

Lemma 2 *Assumption (CC) is satisfied if a dense collection of uniform distributions on T_j can be induced by $\{\mathcal{S}_j\}$, or a dense collection of (truncated) normal distributions on T_j can be induced by $\{\mathcal{S}_j\}$, $j \in E \setminus \{i\}$.*

Proof: See Appendix.

Note that the examples provided in Lemma 2 are merely sufficient and may not be necessary for generating L^∞-complete classes. More complete classes can be found along the same line as in the proof of Lemma 2. For example, one can show that as long as a signal system can generate distributions that approximate uniform distributions in L^1 on T, then such signal system also induces a complete class for L^∞.[19]

By Assumption (CC) and the interim equivalence result stated in Lemma 1, it is immediate that the payment rule in any optimal DRM must be essentially a VCG payment rule. Before we give a formal proof for the uniqueness result, we need one more definition, that is, the class of Vickrey-equivalent auctions.

Definition 3 *(Vickrey-equivalent auctions:)* An auction is a Vickrey-equivalent auction if it is a dominant strategy mechanism in which the bidder with the highest valuation wins and pays the amount of the second-highest valuation,

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[19]Theoretically speaking we can always identify the dual space as one of the complete classes. However, the dual of L^∞ is (strictly) larger than L^1, which does not contain a countable base. Given the difficulty in characterizing such a dual space, we turn to a different approach to look for alternative complete classes, which results in Lemma 2.

and all the other bidders pay zero.[20]

For example, let $\varphi(\cdot)$ be any nonnegative and strictly increasing function, and $b_{2;n}$ be the second highest bid; then any auction in which the bidder with the highest bid wins and pays $\varphi(b_{2;n})$ is a Vickrey-equivalent auction.

Lemma 3 *In any Vickrey-equivalent auction, if we eliminate strategies that nobody ever plays, the resulting restricted mechanism is strategically equivalent to a Vickrey auction.*

Proof: See Appendix.

We are finally ready to state the uniqueness result:

Theorem 2 *Under Assumption (CC), any optimal (sealed-bid) auction must be (essentially) a Vickrey auction, up to its equivalent class.*

Proof: By Lemma 1, any optimal DRM must be interim equivalent to a VCG mechanism. By Assumption (CC), any optimal DRM must be (essentially) a VCG mechanism. Therefore any optimal auction must be (essentially) a dominant strategy mechanism in which the bidder with the highest valuation wins and pays the amount equal to the second highest valuation, and all the other bidders do not pay. By Definition 3, any optimal auction must be (essentially) a Vickrey-equivalent auction. *Q.E.D.*

In view of Theorem 1 and Theorem 2, the Vickrey auction (up to its equivalent class) with free entry, is the unique optimal sealed-bid auction in this costly entry environment consistent with Definitions 1-3 and Assumption (CC).

As is clear by now, the driving force for this uniqueness result is the allocation efficiency. Due to endogenous entry, any optimal auction must induce (ex post) efficient allocation. Since the bidders can update their beliefs after entry so that the final auction is typically an asymmetric one, the only auction that guarantees the efficient allocation is the Vickrey-equivalent auction, regardless of the post-entry environment. In other words, the only auction that survives "environment test" is the Vickrey auction, up to its equivalent class.

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[20]By definition, a Vickrey auction is trivially a Vickrey-equivalent auction.

5 Discussion

An alternative proof of the uniqueness result can be constructed by combining the complete class assumption and a result proven in Holmström (1979). First, using a stronger version of the complete class assumption,[21] any IC DRM must use dominant strategies. Second, we apply Holmström's result to conclude that any dominant strategy DRM implementing an efficient outcome must be a Groves mechanism. Specifically, Holmström shows that when the agent's valuation domain is smoothly connected (in particular, convex), then any payment rule implementing an efficient outcome in dominant strategies is a Groves scheme. In our case, each bidder's valuation function does have a convex domain, thus Holmström's result applies and the optimal DRM must be a VCG mechanism. Note that this alternative proof and the proof given in the previous section are essentially the same in spirit.

Similarly to Holmström's approach, earlier papers by Green and Laffont (1977,1978), and Walker (1978) also identify conditions on the utility domain under which the payment rule implementing an efficient outcome in dominant strategies must be a Groves scheme. In this paper, we introduce an alternative approach leading to a Groves mechanism: instead of identifying conditions on the utility domain, we focus on conditions concerning the information system.

We are now ready to point out the role of restricting our attention to standard auctions in our analysis. It turns out that if such restriction is not imposed, the optimal auction will not be unique. Consider the following mechanisms in which the payment rules are contingent on s: If $s_1, \cdots, s_n$ are the same (so that bidders are symmetric in terms of the beliefs about one another's valuations), then the highest bidder wins and pays what she bids (first-price auction); if $s_1, \cdots, s_n$ are not the same (so that the bidders are typically asymmetric in terms of the beliefs about one another's valuations), then the highest bidder wins and pays the amount of the second-highest bid (second-price auction). Since the payment rule of such a mechanism is contingent on s, the proposed auction is not a standard auction considered in this research. Nevertheless this s-contingent auction mechanism is optimal, as it can achieve ex post efficient allocation regardless of the post-entry environment. The problem with the above example, however, is that the seller needs to conduct an intermediate mechanism to elicit the truthful report on s before the final auction is conducted (otherwise the rules of the final auction cannot be made contingent

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[21]That is, $\mathcal{H}$ forms a complete class for $\mathcal{F}$ if $\forall f, g \in \mathcal{F}$, $\int fh \geq \int gh$ for every $h \in \mathcal{H}$, $h \geq 0$ implies that $f \geq g$ a.e. In the appendix we actually show that Lemma 2 holds for this stronger version of complete classes.

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on s).[22]   Our restriction to standard auctions is thus equivalent to the assumption that the seller cannot conduct an intermediate-stage mechanism to learn s, or conducting such an intermediate mechanism is costly.[23] Note that Bergemann and Morris (2004) do not consider prior-contingent mechanisms either, which is equivalent to our approach to explicitly restrict our attention to standard auctions.[24]

In the preceding analysis we restrict our attention to symmetric entry equilibrium. But as pointed out in Footnote 10 there are asymmetric entry equilibria as well. Such equilibria arise more naturally if the entry is sequential. That is, if potential bidders enter the auction sequentially so that a bidder enters the auction if and only if doing so yields an nonnegative expected profit conditional on the assumption that she would be the last to enter.[25] Given the selling mechanism, the induced equilibrium number of entrants and the equilibrium expected revenues are exactly the same in both the simultaneous and sequential entry cases. Thus in what follows we will not make distinction between these two cases and will simply refer both of them as deterministic entry (asymmetric entry), as opposed to the stochastic entry (symmetric entry) that we analyzed in the previous sections.

It turns out that our main results are still valid after modifying the statement regarding the optimal entry fee.   Let n^* be the number of entrants induced by free entry under a Vickrey auction.   Due to the integer problem the entrants will usually end up with strictly positive expected profit.   Let e^* be the entry fee (imposed to each entrant) that is needed to extract the remaining rent. Then we can show that the Vickrey auction with entry fee e^* is the optimal auction (the proof is analogous to the proof of Theorem 1, and

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[22]Since the "costs" of eliciting reports on s discussed in Footnote 9 are not modeled in this paper, we allow for the possibility that the seller can elicit truthful reports on s without incentive problems.

[23]If conducting an intermediate mechanism to learn s is costly for the seller, the seller will optimally choose not to learn s given the optimality of Vickrey auction. Thus restricting to auctions not contingent on s is without loss of generality in the search for optimal auctions.

[24]If Bergemann and Morris allow for intermediate stages to elicit reports on true priors and consider prior-contingent mechanisms, then the equivalence problem addressed in their paper may become less interesting. For example, in many settings prior-contingent mechanisms may achieve efficiency through either Bayesian implementation or dominant strategy implementation (depending on the true priors, like the example we constructed above). But if that is the case the "equivalence" between Bayesian and dominant strategy implementations becomes not even well defined.

[25]By symmetry, the expected profit to each entrant bidder is the same and no one who has entered will have incentive to leave later.

the optimality of entry fee e^* can be shown by adapting the arguments from Engelbrecht-Wiggans (1993), who analyze asymmetric entry under a second-price auction). Since the seller still cares the full efficiency, the result about the uniqueness of optimal auctions also remains unchanged.

But the expected revenue generated from the optimal auction with deterministic entry will be higher than in the symmetric entry counterpart. To see this, define $S(n)$ to be the equilibrium expected total surplus generated from a Vickrey auction conditional on the number of entrants n. It can be shown that $S(n)$ is concave in n.[26] Now consider an optimal auction with symmetric entry. Without loss of generality we assume that the expected number of entrants in equilibrium, En, is an integer. Define a mean-preserving mechanism to be a Vickrey auction in which the entry fee is set so that bidders are just willing to participate, and exactly En bidders enter the auction. By Jensen's inequality, we conclude that the optimal auction with symmetric entry is revenue dominated by the mean-preserving mechanism constructed above. Since the Vickrey auction with entry fee e^* is optimal among all mechanisms with asymmetric entry (including the mean-preserving mechanism), this in turn implies that the optimal auction with symmetric entry is revenue dominated by the optimal auction with asymmetric entry. Thus if the seller can influence the entry process, he would prefer deterministic entry (asymmetric entry) to stochastic entry (symmetric entry).[27]

Finally it is worth noting that the bidders' ex ante symmetry is also crucial for the uniqueness result to hold. A counter-example can be adapted from Landsberger *et al.* (2001). In their paper, a first-price sealed bid auction with two bidders is analyzed when the ranking of valuations is common knowledge among bidders. They show that there exists an equilibrium in which each bidder bids according to a strictly increasing function, with the lower bidder bidding more aggressively than the higher bidder. As a result, the lower bidder wins the object with some positive probability. Now introduce entry cost into the model so that bidders have to incur a cost c in order to learn their exact valuations, but suppose the ranking of the valuations is still common knowledge before entry. Given the result in Landsberger *et al.* (2001), under a first-price sealed-bid auction, it is obvious that if the entry cost is not too

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[26]Since the Vickrey auction is efficient, we have $S(n) = E(t_{1;n}|n) - nc$, which can be verified to be concave in n.

[27]If En is not an integer, then we can consider the following modified mean-preserving mechanism: with probability $\theta = [En] + 1 - En$, $[En]$ bidders enter the auction, and with probability $1 - \theta = En - [En]$, $[En] + 1$ bidders enter the auction. In either case, entry fee is set to keep bidders being just willing to enter (making zero profit). It is easily seen that all the arguments still go through.

large, both bidders will still participate in the auction and the expected revenue will be positive. However, if a Vickrey auction is conducted, the lower bidder will never enter as long as the entry cost is positive (no matter how small it is), since she would never win the object and the entry cost could never be compensated in equilibrium.[28] As a result, the auction will be conducted in the presence of the higher bidder only. This example can be easily extended to the n-bidder case. That is, only the highest bidder will enter a Vickrey auction, leading to zero revenue for the seller. In this case, the Vickrey auction is not revenue maximizing, but rather revenue minimizing! This example shows how sensitive our result is to the environment.[29]

6   Conclusion

Optimality and efficiency are central to both auction design and practice. While closely related, optimality and efficiency do not always coincide. Revenue maximization typically involves a trade-off between efficiency and rent extraction, and hence is technically complicated and could even be informationally infeasible. In the symmetric entry equilibrium, these two goals coincide. This greatly simplifies the job faced by a mechanism designer. Since any mechanism achieving optimality must award the good to the bidder with the highest valuation, the mechanism designer can restrict the search for optimal auctions to the class of efficient mechanisms. Taking entry into account appears to complicate the setup at the outset, but the optimal auction design problem turns out to be simpler in the sense that a simple auction itself is optimal.

There are several reasons why the Vickrey auction is appealing.[30]   For example, under a Vickrey auction, truthful reporting is induced as a dominant strategy regardless of the beliefs bidders possess about one another's private values, the auctioneer may feel fairly confident that a rational bidder will indeed play the dominant strategy, and the outcome is ex post efficient in dominant strategy equilibrium. This research suggests one more reason to

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[28]Again, assume that both bidders use their dominant strategies and bid their values truthfully.

[29]Besides the ex ante symmetry, bidders' payoff structure also matters for the optimality of Vickrey auctions. For example, Vickrey auctions may fail to be revenue-maximizing if values are interdependent, as efficiency may not be guaranteed in such an environment even under a Vickrey auction.

[30]Due to the (outcome) equivalence between a Vickrey auction and an English auction in private value environment, the following statement also applies to English auctions.

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appreciate the Vickrey auction, that is, within the whole class of auctions that we consider, the Vickrey auction, up to its equivalent class, is not only optimal, but also the only optimal auction if the information potentially available to bidders after entry is sufficiently rich.

Our research suggests one new motivation for looking for robust mechanisms design. Besides, our research provides a new approach to derive conditions under which Bayesian implementation leads to dominant strategy implementation. It has long been conjectured that the conditions identified in Bergemann and Morris (2004) are merely sufficient, and some tighter conditions can be found. The complete class condition proposed in this paper, can be viewed as one such tighter condition, though the "tightening" is more technical rather than about economics.

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Appendix

Proof of Lemma 2: Let $\mathcal{H}$ be a family of nonnegative functions on T. That $\mathcal{H}$ forms an L^∞- complete class is equivalent to the following statement: $\forall f, g \in L^\infty$, $\int fh = \int gh$ for any $h \in \mathcal{H} \Rightarrow f = g$ a.e. By the linearity of integration, $\mathcal{H}$ forms an L^∞-complete class if $\int fh = 0$ for any $h \in \mathcal{H} \Rightarrow f = 0$ a.e. Since $h \geq 0$, to show that $\mathcal{H}$ is an L^∞-complete class, it suffices to show that $\int fh \geq 0$ for all $h \in \mathcal{H} \Rightarrow f \geq 0$ a.e. Let $d = |E| - 1$, where $|E|$ denotes the number of entrant bidders. The proof will be completed by establishing the following five steps.

Step 1: The class (denoted as $\mathcal{H}1$) of all indicator (or characteristic) functions 1_A, where A is some measurable subset of T_{-i} and $0 < m(A) < \infty$, forms a complete class for $L^\infty(T_{-i})$.

Suppose not, then $\int fh \geq 0$ for all $h \in \mathcal{H}1$, but $m(\{f < 0\}) > 0$. Since $\{f < 0\} = \cup_{n=1}^\infty \{f < -\frac{1}{n}\}$, this implies that $m(\{f < -\frac{1}{n^*}\}) > 0$ for some n^*. Now choose a subset A, $A \subset \{f < -\frac{1}{n^*}\} \subset T_{-i}$, such that $0 < m(A) < \infty$. Let $h = 1_A$. Obviously $h \in \mathcal{H}1$, but $\int fh = \int_A f < -\frac{1}{n^*}m(A) < 0$, a contradiction.

Step 2: The class (denoted as $\mathcal{H}2$) of all indicator functions 1_B, where B is some d-dimensional "rectangle" in T_{-i}, forms a complete class for $L^\infty(T_{-i})$.

Given any indicator function 1_A, where A is a subset of T_{-i} defined in Step 1, it can be approximated by step functions on T_{-i} in L^1. Hence,

$$\sum_{i=1}^n c_i 1_{B_i} \to 1_A \text{ in } L^1 \tag{13}$$

for some c_i and B_i, where $c_i > 0$ and B_i's are disjoint d-dimensional "rectangles."

Therefore,

$$
\begin{aligned}
|\int f \sum c_i 1_{B_i} - \int f 1_A| \quad &\leq \quad \int |f(\sum c_i 1_{B_i} - 1_A)| \\
&\leq \quad \|f\|_\infty \cdot \|\sum c_i 1_{B_i} - 1_A\|_1 \text{ (by Hölder's Inequality)} \\
&\to \quad 0 \quad \text{(by (13) and that } f \in L^\infty)
\end{aligned}
$$

On the other hand, we have

$$\int f \sum c_i 1_{B_i} = \sum c_i \int f 1_{B_i} \geq 0 \text{ (by hypothesis)}$$

Therefore, $\int f 1_A \geq 0$. Since 1_A is an arbitrary indicator function in $\mathcal{H}1$, we must have $f \geq 0$ a.e. by the conclusion in Step 1. This implies that $\mathcal{H}2$ is a complete class for L^∞.

Step 3: The class (denoted as $\mathcal{H}3$) of all functions $\prod_{j \in E \setminus \{i\}} u_j$ on T_{-i}, where u_j is some uniform density function defined on T_j, is a complete class for $L^\infty(T_{-i})$.

This step follows from Step 2 trivially since any function 1_B defined in Step 2 can be normalized to be a product of d uniform density functions.

Note that $\mathcal{H}3$ can do the job as long as it contains a dense collection of the uniform density functions described above. Moreover, in our model, the joint density function on t_{-i} induced by $\{\mathcal{S}_{-i}\}$ is the product of $|E| - 1$ one-dimensional density functions (by conditional independence). This completes the proof for the first part of the claim in Lemma 2.

Step 4: Any one-dimensional uniform density function on T_j can be approximated in L^1 by a finite mixture of (truncated) normal density functions on T_j.

First, it is well-known that the approximation can be achieved by a finite mixture of normal density functions on the real line.[31] Specifically, given any uniform density function $h(x) = \frac{1}{b-a} 1_{(a,b)}(x)$, where $(a, b) \subset T_j$, it can be approximated in L^1 by the equi-weighted mixture of n normal densities over R, with means $\mu_{n,k} = (b - a)\frac{k-0.5}{n} + a$ for $k = 1, \cdots, n$, and common standard deviation $\sigma_n = \frac{1}{n}$. Thus if we define

$$h_n(x) = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{2\pi}\sigma_n} \cdot e^{-\frac{(x-\mu_{n,k})^2}{2\sigma_n^2}},$$

we have $h_n(x) \to h(x)$ in L^1 for any $x \in R$. We next show that the approximation can be achieved by a finite mixture of truncated normal density functions on T_j.

Define

$$C_{n,k} = \int_{T_j} \frac{1}{\sqrt{2\pi}\sigma_n} \cdot e^{-\frac{(x-\mu_{n,k})^2}{2\sigma_n^2}} \, dx$$

Then the truncated normal density function of $N(\mu_{n,k}, \sigma_n^2)$ on T_j is given by

$$\varphi_{n,k}(x) = \frac{1}{C_{n,k}} \frac{1}{\sqrt{2\pi}\sigma_n} \cdot e^{-\frac{(x-\mu_{n,k})^2}{2\sigma_n^2}} \tag{14}$$

Let $C_n = \int_{T_j} h_n(x) \, dx$, then

$$C_n = \frac{1}{n} \sum_{k=1}^{n} C_{n,k}$$

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[31]For example, this result is mentioned in the introduction of Ferguson (1983).

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Since T_j is a subset of R, that $\int_R |h_n(x) - h(x)|\, dx \to 0$ implies that $\int_{T_j} |h_n(x) - h(x)|\, dx \to 0$, which in turn implies $\int_{T_j} h_n(x)\, dx \to \int_{T_j} h(x)\, dx = 1$, i.e., $C_n \to 1$.

Define new weights $\omega_{n,k} = C_{n,k}/(nC_n)$, and form a mixture of truncated normal densities:

$$
\begin{aligned}
g_n(x) &= \sum_{k=1}^{n} \omega_{n,k}\varphi_{n,k}(x) \\
&= \sum_{k=1}^{n} \frac{C_{n,k}}{nC_n}\frac{1}{C_{n,k}}\frac{1}{\sqrt{2\pi}\sigma_n}\cdot e^{-\frac{(x - \mu_{n,k})^2}{2\sigma_n^2}} \\
&= \frac{1}{C_n}\cdot h_n(x)
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{T_j} |g_n(x) - h(x)| &\leq \int_{T_j} |g_n(x) - h_n(x)| + \int_{T_j} |h_n(x) - h(x)| \\
&= |\frac{1}{C_n} - 1|\int_{T_j} h_n(x) + \int_{T_j} |h_n(x) - h(x)| \\
&\to 0.
\end{aligned}
$$

This completes the proof for Step 4.

Step 5: The class (denoted as $\mathcal{H}4$) consisting of all functions $\{\prod_{j \in E \setminus \{i\}} \varphi_j(x_j)\}$, where $\varphi_j(\cdot)$ is some truncated normal density function on T_j defined in (14) is a complete class for $L^\infty(T_{-i})$.

We show a lemma first.

Lemma A *Let u_j be a uniform density function on T_j, then given any function $\prod_{j \in E \setminus \{i\}} u_j$, there exists $\{\Psi_{n_j}^{(j)}\}_{j \in E \setminus \{i\}}$, where each $\Psi_{n_j}^{(j)}$ is a mixture of n_j truncated normal densities on T_j, such that $\prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)}(x_j) \to \prod_{j \in E \setminus \{i\}} u_j(x_j)$ in L^1 as $n_j \to \infty$ for all $j \in E \setminus \{i\}$.*

Proof: Let's consider the case $d = 2$. As shown in Step 4, given any two one-dimensional uniform density functions $u_1(x_1)$ and $u_2(x_2)$, there exist two finite mixtures of (truncated) normal density functions $\Psi_{n_1}(x_1)$ on T_1 and $\Psi_{n_2}(x_2)$ on T_2 such that $\|\Psi_{n_j} - u_j\|_1 \to 0$ for $j = 1, 2$.

$$
\begin{aligned}
&\int |\Psi_{n_1}(x_1)\Psi_{n_2}(x_2) - u_1(x_1)u_2(x_2)|\, dx_1\, dx_2 \\
\leq\ &\int [|\Psi_{n_1}(x_1) - u_1(x_1)|\Psi_{n_2}(x_2) + u_1(x_1)|\Psi_{n_2}(x_2) - u_2(x_2)|]\, dx_1\, dx_2 \\
\leq\ &\int [|\Psi_{n_1}(x_1) - u_1(x_1)| \cdot (|\Psi_{n_2}(x_2) - u_2(x_2)| + u_2(x_2))
\end{aligned}
$$

$$+ u_1(x_1) |\Psi_{n_2}(x_2) - u_2(x_2)|] \, dx_1 \, dx_2$$
$$= \int |\Psi_{n_1}(x_1) - u_1(x_1)| \, dx_1 \cdot \int (|\Psi_{n_2}(x_2) - u_2(x_2)| + u_2(x_2)) \, dx_2$$
$$+ \int u_1(x_1) \, dx_1 \cdot \int |\Psi_{n_2}(x_2) - u_2(x_2)| \, dx_2$$
$$\rightarrow \quad 0$$

Using the method of mathematical induction, the generalization to an arbitrary d-dimensional case is straightforward. *Q.E.D.*

Now given any $h \in \mathcal{H}3$, by Lemma A, we can find mixtures of (truncated) normal densities $\{\Psi_{n_j}^{(j)}\}_{j \in E \setminus \{i\}}$ such that $\prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)} \rightarrow h$ in L^1. By Hölder's Inequality again, we have

$$\int f \cdot \prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)} \rightarrow \int fh \tag{15}$$

On the other hand,

$$\int f \cdot \prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)}(x_j) = \int f \cdot \prod_{j \in E \setminus \{i\}} \left(\sum_{k=1}^{n_j} \omega_{n_j,k} \varphi_{n_{j,k}}^{(j)}(x_j) \right)$$
$$= \int f \cdot \sum_{k=1}^{\prod_{j \in E \setminus \{i\}} n_j} \omega_k \Omega_k$$
$$= \sum_{k=1}^{\prod_{j \in E \setminus \{i\}} n_j} \omega_k \int f \Omega_k$$
$$\geq 0 \quad \left(\text{since } \int f \Omega_k \geq 0 \text{ by hypothesis} \right) \tag{16}$$

In the above expressions Ω_k is a product of d (truncated) normal densities and ω_k is a product of d non-negative weights. By (15) and (16),

$$\int fh \geq 0$$

This holds for arbitrary $h \in \mathcal{H}3$, by the conclusion in Step 3, $f \geq 0$ *a.e.*, which implies that $\mathcal{H}4$ is a complete class for L^∞.

This completes the proof for the second part of the claim in Lemma 2. *Q.E.D.*

Proof of Lemma 3: Let $\Gamma(W, g(\cdot))$ be the resulted mechanism from a Vickrey-equivalent auction, after eliminating strategies that nobody use, where

W is the resulted strategy set, and $g(\cdot)$ is the outcome function, mapping a bid profile to an assignment decision and payments. To show that Γ is strategically equivalent to a Vickrey auction, we need to show that there exists an isomorphism (a one-to-one mapping) between W and T which preserves payoffs.

Define the social choice function $f(t) = (y(t), x(t))$ where $y(t)$ is an efficient assignment rule and $x(t)$ is the Vickrey payment rule. By Definition 3, there is a dominant strategy $B^*(\cdot)$ in mechanism Γ that implements $f(t)$. We therefore have

$$g(B^*(t)) = f(t) \tag{17}$$

where $B^*(t) = (b_1^*(t_1), b_2^*(t_2), \cdots, b_n^*(t_n)) \in W$.

First, Definitions 1 and 3 imply that $B^*(\cdot)$ has to be symmetric and strictly increasing; otherwise the outcome may not be efficient for some realizations of t. Therefore $b_i^*(\cdot) = b^*(\cdot) \ \forall \ i \in E$ and $b^*(\cdot)$ is strictly increasing. Hence $B^* : T \to W$ is a one-to-one mapping.

It remains to show that the mapping B^* is payoff-preserving. Given a report profile $\hat{t} = (\hat{t}_i, \hat{t}_{-i}) \in T$, i's utility derived from a Vickrey auction can be written as $V_i(\hat{t}; t_i) = u_i(f(\hat{t}); t_i)$. Given a report profile $B^*(\hat{t}) \in W$, bidder i's utility derived from mechanism Γ is given by

$$
\begin{aligned}
\hat{V}_i(B^*(\hat{t}); t_i) &= u_i(g(B^*(\hat{t})); t_i) \\
&= u_i(f(\hat{t}); t_i) \quad \text{(by (17))} \\
&= V_i(\hat{t}; t_i)
\end{aligned}
$$

Therefore, the one-to-one mapping B^* is indeed payoff-preserving. $Q.E.D.$

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