Available online at www.sciencedirect.com
ScienceDirect
Games and Economic Behavior 58 (2007) 181–207
GAMES and Economic Behavior
www.elsevier.com/locate/geb

# Indicative bidding and a theory of two-stage auctions

Lixin Ye *

*Department of Economics, The Ohio State University, 417 Arps Hall, 1945 North High Street, Columbus, OH 43210, USA*

Received 6 February 2004

Available online 2 February 2006

**Abstract**

Motivated by the practice of indicative bidding, which is commonly used in sales of complex and valuable assets, this paper develops a theory of two-stage auctions based on the assumption that learning valuation is costly. In various cases that characterize what can be learned in the due diligence stage, we show that no symmetric increasing equilibrium exists and hence efficient entry cannot be guaranteed under the current design of indicative bidding. However, by introducing auctions of entry rights with binding first-round bids, we show that efficient entry can be induced under certain conditions. We also show that optimal auctions are typically characterized by a limited number of final bidders, which justifies the general practice of conducting two-stage auctions in environments with costly entry.


*JEL classification:* D44; L10; D82

*Keywords:* Auctions; Two-stage auctions; Indicative bidding; Efficient entry

## 1. Introduction

Indicative bidding is a common practice in sales of business assets with very high values. It works as follows: the auctioneer (usually an investment banker) actively markets the assets through phone calls and mailings to a large group of potentially interested buyers. The auctioneer then asks the potential buyers to submit non-binding bids, which are meant to indicate the buyers' interest in the assets. The auctioneer uses the bids together with other information submitted by

* Fax: (+1) (614) 292 3906.
*E-mail address:* lixinye@econ.ohio-state.edu.

0899-8256/$ – see front matter 
doi:10.1016/j.geb.2005.12.004

the bidders to select a final set of eligible bidders, typically five to ten in number.[1] The auctioneer then communicates only with these final bidders regarding various aspects of the auction (timing, terms of the sale, and so on), provides them with extensive access to information about the assets, and finally runs the auction, typically using binding sealed bids.

This practice of indicative bidding is quite widespread. For example, in the United States in recent years, electrical generating assets worth billions of dollars have been auctioned as part of the restructuring of the industry. For the dozens of cases about which we have detailed information, indicative bidding was used in every case. Examples include Maine's Central Maine Power (CMP), which placed its entire 2110 megawatt (MW) asset portfolio up for auction. A total of 1121 MW were sold in the initial auction at a price of $846 million.[2] In California, Pacific Gas & Electric (PG & E) decided to voluntarily divest virtually all of its fossil generation; as of 1999, assets worth more than $1.5 billion had been auctioned.[3] In Oregon, Portland General Electric (PGE) intends to become a regulated transmission and distribution company and is seeking to sell all its generation and related assets.[4] Each of these examples incorporates a two-stage auction design described above.

The widespread use of indicative bidding is at least initially puzzling. Upon first hearing a description of the practice, many economists ask: "Why don't the potential buyers simply make infinite, or at least very large bids in the indicative stage?" And, "Why would a seller want to limit participation in the auction?"

The answer to the first question is that in the sale of a billion-dollar asset, the final round bidders must often spend millions of dollars to study the assets closely ("due diligence") and prepare a bid.[5] Consequently, rational bidders do not want to be included in the final round unless they have a good chance of winning the auction. This may dissuade bidders from making very high bids at the indicative stage, particularly when there are large numbers of other bidders bidding cautiously. The answer to the second question is that the auctioneer often finds it impossible to attract a large number of bidders without the indicative stage, because bidders do not want to spend millions of dollars participating in an auction against a large number of competitors. Most practitioners believe that indicative bidding increases participation in the auctions.

---

[1] The selection is mainly based on the prices that bidders indicated. The other information such as financial status or industry experience is usually used to establish the bidders' qualification for bidding.

[2] The sale opened in May, 1997. Non-binding bids were due by September 10, 1997. The final round bidders were selected by CMP and its financial adviser, Dillon Read. Phase II bids were due by December 10, 1997. FPL Energy was selected as the final winner to buy the fossil, hydro and biomass packages.

[3] With Morgan Stanley serving as the financial adviser, two auctions were conducted splitting the plants among three buyers. The initial auction began in September, 1997 and concluded in November with Duke Energy being selected as the buyer (at a price of $501 million). The second auction began in April, 1998 and concluded in November, 1998 with Southern Energy being selected to buy the fossil plants (at a price of $801 million), and FPL Energy the geothermal plants (at a price of $213 million).

[4] PGE retained Merrill Lynch to serve as its financial adviser. $230.5 million of assets were auctioned with PP & L Global as the final winner in November, 1998. But the whole divestiture plan has yet to be approved by the Oregon Public Utility Commission.

[5] A bidder will typically have to incur substantial costs to conduct due diligence to prepare a bid, during which process the bidders will usually have to use considerable personnel resources and may have to forgo other purchase opportunities (in order to participate fully in the current process). Due diligence usually involves a multitask team with strong project management skills. For example, due diligence in the sale of a power plant typically includes engineering assessment of assets, determination of future staffing levels, labor commitments and benefits, environmental contamination assessment, financial review of contractual obligations, determination of bid price and bidding strategies, and legal review and drafting, and so on. Vallen and Bullinger (1999) provide a detailed description of one such process.

The above answers are intuitive, but there are other important questions as well: can the indicative bidding stage reliably select the most qualified bidders for the final auction? If not, then what alternative rules can perform better? How exactly does the two-stage auction design alleviate the effect of entry costs on the potential buyers? And, what are the optimal (i.e. revenue maximizing) selling schemes in this context?

To answer all these questions, a formal theory of two-stage auctions with costly entry is needed. This paper is the first attempt at building such a theory. Our main results are as follows: First, efficient entry cannot be guaranteed under the current design of indicative bidding. Second, by introducing auctions for entry rights with binding first-round bidding, the seller may in general induce efficient entry and improve the performance of two-stage auctions. Third, the seller bears all the entry cost in equilibrium, and thus the optimal auction is characterized by a finite number of final bidders. The first result identifies a potential problem with the current design of indicative bidding, the second result suggests an alternative selection mechanism which may outperform the current design of indicative bidding, and the last result justifies the practice of using two-stage procedures in this environment with costly entry.

This research is first related to the literature of auctions with costly entry.[6] In one branch of the literature, bidders are assumed to possess no private information before entry and they learn their types only after entry (e.g., Johnson, 1979; French and McCormick, 1984; McAfee and McMillan, 1987; Engelbrecht-Wiggans, 1993; Levin and Smith, 1994). Samuelson (1985) is an exception to the above formulation. In a model of a competitive procurement auction, he considers the case in which bidders know their private types before entry and entry is merely a process to incur bid-preparation costs without information updating. The model in this paper combines and extends these two formulations in the existing literature. For one, we assume that each bidder is endowed with a private value component before entry; for the other, bidders are allowed to learn another private or common value component after entry. In this sense, this paper presents a more general framework in analyzing auctions with costly entry.

Under this framework, we show that there is no symmetric increasing equilibrium under the current design of indicative bidding. Absent a symmetric increasing equilibrium, efficient entry cannot be guaranteed. In other words, the most qualified bidders cannot be reliably selected for the final sale. Thus the optimality of the sale is questionable under the current practice. Considering the billions of dollars involved in the current practice, the efficiency loss could be potentially substantial.

However, by no means do we suggest abandoning the general idea of two-stage auction design in this context. By modifying the current indicative bidding with alternative rules, we show that efficient entry can be induced under certain conditions. This relates our research to Fullerton and McAfee (1999), who analyze a research tournament model with multiple contestants competing for a common prize. They show that for a large class of contests the optimal number of finalists is two. They then introduce a scheme of auctioning entry into tournaments which can implement efficient entry. Seeking alternative rules that may perform better than the current indicative bidding design, we follow Fullerton and McAfee's lead and also consider schemes of auctioning entry rights so that the first-round bids reflect binding entry fee payments. We show that under certain conditions, either a uniform-price or a discriminatory-price auction can induce

[6] The earlier literature on optimal auction design and revenue comparison (e.g., Vickrey, 1961; Riley and Samuelson, 1981; Myerson, 1981; Milgrom and Weber, 1982) generally assumes that there is an exogenously specified set of bidders and that these bidders are endowed with information about the object's valuation; both potential entry and bid preparation costs are ignored.

efficient entry; while an all-pay auction can always induce efficient entry as long as a sufficiently large entry award is provided. While these results are similar to that in Fullerton and McAfee, our analysis is not a trivial extension. First, our model is quite different from theirs. In particular, while in Fullerton and McAfee complete information is assumed for the post-entry game, in our model incomplete information is preserved both before and after entry. Second, the implication of the optimal number of finalists in our model is quite different from that in Fullerton and McAfee: while this number is always two in their model, we show that it is one when entry does not involve value updating, and can be two or more than two when entry involves value updating.

By introducing auctions for entry rights, our research also relates to the literature of multi-unit auctions. In multi-unit auctions with independent private valuation, the revenue equivalence theorem continues to hold if each bidder demands exactly one unit (e.g., Harris and Raviv, 1981; Weber, 1983; Maskin and Riley, 1989). In our models, entry rights are allocated through the first-stage bidding. Thus the first-stage auction is basically one with multiple units in which each bidder demands exactly one unit. Though the values attached to these multi-units (entry rights) are not as straightforward as in the usual (single-stage) multi-unit auctions, we are nonetheless able to show that in any symmetric increasing equilibrium, expected payoffs are the same under different auction formats. We thus extend a revenue equivalence result to a two-stage auction context.

Motivated by puzzles arising from indicative bidding, this research ends with a theory of two-stage auctions. In one way or another, two-stage auctions are commonly employed in privatization, procurement, takeover, and merger and acquisition contests all over the world. For example, on May 10, 1999, the NTL Broadband Cable Co. announced its acquisition of Cablelink Limited, Ireland's largest cable television provider. The sale was conducted using a two-stage auction in which 5 bidders were selected to participate into the second round. The final price obtained was 535.180 million Irish Pounds (approximately US$730 million). Another example is the sale of Daewoo Motors. Daewoo ran into financial trouble in July, 1999, prompting creditors to provide emergency loans and to start dismantling what was then the country's second largest conglomerate. Indicative bidding was also used in this takeover process.[7] The importance of two-stage auctions is also reflected in the work by Perry et al. (2000) and McAdams et al. (1999). Both papers show that a carefully designed two-stage sealed-bid auction can match the English ascending-bid auction in aggregating information. Our approach in this research differs from theirs in two important aspects. First, we model the indicative bidding as completely non-binding, while in their models the first-stage bids impose some restriction on what can be bid in the second stage: In Perry et al., the first-stage bid becomes the minimal bid in the second stage, and in McAdams et al., the second-stage bids cannot be lower than the highest rejected first-stage bid. Second, in contrast to their assumption of no entry cost, we assume that information acquisition and bidding are costly. We maintain these differences in the modeling since nonbinding first-stage bidding and costly entry are two most prominent features in all our motivating examples. We thus hope that our results can add new insights into the understanding of two-stage auctions.

The paper is organized as follows. Section 2 introduces the model. Section 3 introduces the concept of efficient entry. Section 4 analyzes the second-stage auction, conditional on $n$ most qualified bidders being selected from the first-stage bidding. Section 5 analyzes the first-stage

[7] In the first round which was conducted in June 2000, the world's second largest auto company, Ford Motor, offered $6.9 billion for Daewoo while DaimlerChrysler–Hyundai jointly bid $4.5 billion and GM came in at the lowest price. Consequently, Ford was picked to be the sole priority bidder to proceed onto the final negotiation phase.

bidding, both under the current design of indicative bidding and the alternative schemes. Section 6 provides solved examples to illustrate the optimal number of final bidders in different cases. Section 7 is a discussion, and Section 8 concludes.

## 2. The model

There is a single, indivisible asset for sale to $N$ potentially interested buyers (firms). Information is revealed in two stages. In the first stage, each potential buyer is endowed with a private value component $X_i$. The $X_i$'s are independent draws from a distribution with CDF $F(\cdot)$. Bidder $i$ knows its own $X_i$ but not its competitors' (denoted as $X_{-i}$). In the second stage, by incurring an entry cost $c$, each entrant bidder learns another signal $Y_i$. Three cases will be analyzed in this paper. The first is *entry without value updating* (or *the pure entry case*), where $Y = 0$.[8] The second case is *entry with private value updating*, where $Y_i$ is another private value component. In this case the total value for entrant bidder $i$ is $X_i + Y_i$. The third case is *entry with common value updating*, where $Y_i$ is a private signal about a common value component. Specifically, the total value of the asset for entrant $i$ is $X_i + \frac{1}{N}\sum_{j=1}^{N} Y_j$.[9] In the entry with private or common value updating cases, the $Y_i$'s are independent draws from a distribution with CDF $G(\cdot)$, which has nonnegative support.

We thus assume that each buyer's signal consists of two parts, one received before entry occurs and the other received after the entry occurs. Moreover, we assume that the two parts of a buyer's signal are additive and all signals are independent. This formulation is somewhat special, but it captures one key feature in sales of highly complicated asset with costly entry. That is, a buyer usually does not know the total value of the asset in the first stage, and the remaining information about the value can only be learned after due diligence process is completed. In the above formulations, $X$ can be interpreted as the valuation about the asset estimated based on each potential buyer's private attributes such as operating experiences, managerial skills, environmental stewardship, labor relations, market power, and corporate citizenship etc.[10] The second-stage signal $Y$ may reflect information containing detailed engineering assessment of the asset, environmental conditions, labor commitments, and supply contracts etc., which may only be learned during the due diligence process. In Section 7, we will consider alternative formulations and demonstrate that our current formulation is not as restrictive as it appears. All major results in this paper appear likely to extend to more general models.

[8] In this case, we can think of the due diligence stage as simply a way to meet some legal requirements.

[9] In traditional common value models, the common value has some known prior distribution and bidders' signals are draws conditional on the underlying common value (see, for example, Rothkopf, 1969, and Wilson, 1977). We adopt an alternative formulation here, i.e., the common value is specified as the average of the bidders' signals, following the lead of Bikhchandani and Riley (1991), Krishna and Morgan (1997), and in particular, Goeree and Offerman (2003). As pointed out in Goeree and Offerman, these two formulations have the same qualitative features. First, the object for sale is worth the same to all bidders. Second, in both formulations bidders should realize that winning means that their signals are likely to be too optimistic (the winner's curse). As $N$ increases, the difference between these two formulations disappears by the law of large numbers. Similarly to our treatment here, in Klemperer (1998), Bulow et al. (1999), and Bulow and Klemperer (2002), the common value is formulated as the sum of the bidders' private signals.

[10] We suppress the common value component in the first-stage signal formulation, as the common value component does not contribute to the difference in bidders' qualifications.

The sale of the asset proceeds in two stages. Bids in the first stage can be binding or non-binding. In the current practice of indicative bidding, the first-stage bids are non-binding.[11] Under the alternative schemes the first-stage bids are binding, indicating firm commitments to pay for entry. In this paper we consider three specific schemes: all-pay auction, uniform-price auction, and discriminatory-price auction. Under an all-pay auction, all bidders need to pay what they bid, regardless of whether or not they are selected. Under a uniform-price auction, all the selected bidders pay an entry fee equal to the highest rejected bid, and bidders who are not selected do not pay. Under a discriminatory-price auction, all the selected bidders pay what they bid and those not selected do not pay. Uniform-price and discriminatory-price auctions are also called *customary auctions*.

Based on the first-stage bids, $n$ bidders with the highest $n$ bids are selected to the second stage, where $n$ is pre-announced by the auctioneer at the outset of the game. The $(n+1)$st highest first-stage bid, or the highest rejected bid, is revealed to the $n$ entrants.[12] Each entrant incurs a cost $c$ to learn $Y_i$. Then they engage in the final sealed-bid auction.

To examine the mechanism of entry cost "adjustment," we consider an entry award (an interim payoff) $K$ in a general selling scheme: when a bidder is selected for entry, she automatically "wins" an entry award $K$. Under the current indicative bidding mechanism, no entry award is offered, so $K = 0$. More generally, $K$ can be negative, which denotes a "tax" on entry. When $K$ is positive, we can also interpret $K$ in terms of the entry subsidy level: $K = c$ implies full subsidization, $K > c$ implies over-subsidization, and $0 < K < c$ implies partial subsidization.

We will maintain a set of assumptions throughout the analysis. First, the seller and the bidders are assumed to be risk-neutral. Second, $X$ and $Y$ are independent, and the density function of $X$, $f(\cdot)$ is continuous and strictly positive on its compact support $[0, \bar{x}]$. Third, the seller's reserve value of the asset is normalized to be zero. Fourth, we assume that the number of potential bidders $N$, or the entry cost $c$, is large enough such that when all $N$ potential bidders enter the auction, some bidder's expected profit will be strictly negative.[13]

To further simplify the analysis, we make two additional assumptions. First, $X$'s density function $f(\cdot)$ is log-concave. This technical condition will ensure the existence of a symmetric increasing equilibrium in the common value updating case, though it is not needed for the analysis of private value updating case. This assumption is satisfied by almost all the commonly used distributions such as uniform, normal, exponential, chi-square, etc. Second, we assume that the auctioneer is unable to set a reserve price in the final auction. A special case is that the auctioneer cannot commit to a reserve price. The restriction of this assumption is discussed in Section 7.

## 3. Efficient entry

In this research we focus on efficient entry exclusively. Note that the usual efficiency criteria (e.g., the ex post efficiency) are not well defined in our context, and it would only be meaningful

[11] Note that indicative bids are not cheap talk, because they will affect the costs the bidders incur and their ability to bid again in the second stage.

[12] In practice there are variations regarding the revelation policy. In our model we consider the policy of revealing the highest rejected bid so that each final bidder enters the second stage with identical beliefs about the first-stage observations of other final bidders. In Perry et al. (2000) and McAdams et al. (1999), they both assume that all losing bids are revealed to the final stage bidders.

[13] If every bidder earns a positive expected profit when all enter, then our model is no different from the traditional models that do not consider entry costs.

to consider efficiencies given the set of entrant bidders. So any sensible measure of efficiencies in this context should include a measure of entry efficiency. More importantly, it is quite obvious that if entry is inefficient, the optimality of the sale cannot be guaranteed. For these reasons we focus on efficient entry exclusively, an approach adopted previously by Fullerton and McAfee (1999).

Simply put, efficient entry means that the most "qualified" buyers enter the final-stage auction. In our model, the most qualified buyers are those with the highest first-stage signals. Given the number of bidders to be selected, we can define efficient entry as follows:

**Definition 1** *($n$-Efficient Entry).* The outcome in which bidders with the $n$ highest first-stage signals ($X_i$'s) are selected for the second-stage auction.

The importance of $n$-efficient entry can be easily seen in our valuation formulations. In all three cases, the first-stage signal ($X$) enters the total valuation function additively. So the higher the $X_i$, the more likely that bidder $i$ will end up with higher total value in the sense of first order stochastic dominance. Therefore, given that the number of finalists is $n$, those bidders with $n$ highest first-stage signals should be selected. If this is not the case, then the optimality of the sale cannot be guaranteed.[14] We thus focus exclusively on $n$-efficient entry in our analysis. Given $n$-efficient entry, we will show that there will be an optimal number of bidders to be selected. Let $n^*$ denote this optimal number of bidders, then efficient entry is $n^*$-efficient entry. For ease of exposition, from now on we will simply refer to efficient entry even when we really mean $n$-efficient entry. Hopefully the distinction is clear within the context.

To guarantee efficient entry given any realization of first-stage signals, the bid function in the first stage must be symmetric and strictly increasing in $x \in [0, \bar{x}]$ (almost everywhere). Note that a weakly increasing bid function is not sufficient to guarantee efficient entry. For brevity of exposition, we refer to *the symmetric increasing equilibrium* as the pure symmetric equilibrium in which each bidder bids according to a strictly increasing first-stage bid function.

In the following analysis, we will work backward, starting with the analysis of second-stage auctions. Since we focus on the case of efficient entry, we will analyze the second-stage auctions conditional on efficient entry. After we derive the equilibrium expected payoffs from the second-stage auctions, we will then analyze the bidding in the first stage.

## 4. The second-stage auctions

We assume that the first-stage bidding induces efficient entry and, to maintain competitive bidding, the number of bidders to be selected is $n \geqslant 2$.[15]

To unify the notation in all three cases, we define a summary statistic $S = X + \lambda Y$, where $\lambda = 0$ in entry without value updating, $\lambda = 1$ in entry with private value updating, and $\lambda = \frac{1}{N}$ in entry with common value updating.

In this paper, we assume that the final auctions conducted in the second stage are standard auctions, which are defined below:

[14] For instance, in the case of pure entry with $n \geqslant 2$, if either the highest bidder or the second highest bidder is left out from the final auction, then the expected revenue will not be maximized under a second-price auction, assuming that bidders employ the dominant strategies to bid their values truthfully.

[15] We will also consider the case $n = 1$ later.

**Definition 2** *(Standard Second-Stage Auctions).* Auctions in which the asset is awarded to the bidder with the highest bid, and the bidder with the lowest possible type (summary statistic) makes zero expected profit.

Let $Z$ be a generic random variable. In what follows, we use $Z_{j;n}$ to denote the $j$th highest value among all $n$ draws of $Z$, and use $Z_{j;-i}$ to denote the $j$th highest value among all draws of $Z$ except $Z_i$ (where $i$, $j$ and $n$ should be interpreted as generic integers here). We also let $\mathbb{N}$ denote the index set of the $N$ first-stage bidders, and $\mathbb{E}$ denote the index set of the $n$ second-stage entrant bidders. Finally we let $\widehat{X}_i$ be the first-stage signal (value) possessed by entrant bidder $i \in \mathbb{E}$, and $\widehat{S}_i = \widehat{X}_i + \lambda Y_i$ be the summary statistic for entrant bidder $i$.

Since the highest rejected first-stage bid is revealed to the second-stage bidders, if a symmetric increasing equilibrium exists in the first-round bidding, then given the highest rejected bid, the first-stage value possessed by the highest rejected bidder, $X_{n+1;N}$, can be inferred by inverting the bid function. The following lemma can be easily verified.

**Lemma 1.** *Conditional on $X_{n+1;N} = x_{n+1}$, the $\widehat{X}_i$'s are i.i.d. with density function $\frac{f(\cdot)}{1-F(x_{n+1})}$.*

So given efficient entry, the second-stage auction is a symmetric auction with independent private signals (in all three cases). The payoff-equivalence result holds trivially in the pure entry and private value updating cases and the bidder with the highest summary statistic or total value $(\widehat{S})$ wins with expected profit equal to $E\widehat{S}_{1;n} - E\widehat{S}_{2;n}$.

In common value updating case, each bidder obtains a private signal about a common value component after entry. Therefore each entrant in the second-stage auction possesses a two-dimensional signal: a private value signal and a common value signal. Given efficient entry, Lemma 1, and the log-concavity assumption regarding $f(x)$, the valuation setup in this case resembles Goeree and Offerman's (2003), with the only difference being that the two-dimensional signals are obtained sequentially, while in Goeree and Offerman they are obtained simultaneously.

Following Goeree and Offerman, a unique symmetric increasing equilibrium can be characterized in terms of the summary statistic $X + \frac{1}{N}Y$ under either a second-price auction or a first-price auction.[16] Under a second-price auction, the unique symmetric increasing equilibrium bid function is given as follows:

$$\begin{aligned} B(s_i) &= E\biggl[s_i + \frac{1}{N}\sum_{l\in\mathbb{N}\setminus\{i\}} Y_l \;\Big|\; \widehat{S}_{1;-i} = s_i\biggr] \\ &= s_i + \frac{N-n}{N}EY + \frac{n-1}{N}E\bigl[Y_j \mid \widehat{S}_{1;-i} = s_i\bigr] \end{aligned} \tag{1}$$

where $j \in \mathbb{E} \setminus \{i\}$.

In equilibrium one's bid represents the asset's expected value to her assuming that she is the "marginal buyer," i.e., when she possesses the minimal summary statistic that is required to win the asset.

[16] In fact, we can show that the summary statistic defined in this way is the only possible summary statistic for ranking the bidder types.

Under a first-price auction, the unique symmetric increasing equilibrium bid function is as follows:

$$B(s_i) = E\left[s_i + \frac{1}{N}\sum_{l\in\mathbb{N}\setminus\{i\}} Y_l \mid \widehat{S}_{1;n} = s_i\right] - E\left[\widehat{S}_{1;n} - \widehat{S}_{1;-i} \mid \widehat{S}_{1;n} = s_i\right]. \tag{2}$$

The first term on the right hand side represents the expected value of the asset to bidder $i$, assuming that her summary statistic is the highest among the $n$ entrants. The second term reflects the winner's informational rent from attending the second-stage auction.

The log-concavity of $f(\cdot)$ ensures that the equilibrium bidding functions (1) and (2) are both strictly increasing.

It can be easily verified that payoff equivalence also follows. In equilibrium, the expected total surplus is $E\widehat{S}_{1;n} + \frac{N-1}{N}EY$ and the expected revenue is $E\widehat{S}_{2;n} + \frac{N-1}{N}EY$. Again, the entrant with the highest summary statistic wins the asset and the expected winner's profit is $E\widehat{S}_{1;n} - E\widehat{S}_{2;n}$.[17]

In terms of the summary statistics defined for all the three cases, we can summarize the results in this section as follows:

**Proposition 1.** *Given efficient entry, in the second-stage* (*standard*) *auctions payoff equivalence holds and the bidder with the highest summary statistic wins the asset with expected profit* $E\widehat{S}_{1;n} - E\widehat{S}_{2;n}$.

In light of this result, the analysis of the first-stage bidding can be restricted to the analysis of a direct revelation mechanism (DRM). This can be done by replacing the second-stage auction with its correlated equilibrium payoffs.

## 5. The first-stage bidding

Let $\mathcal{F}$ denote the format of the first-stage auction, where $\mathcal{F} \in \{I, A, U, D\}$, and $I, A, U, D$ denote nonbinding indicative bidding, all-pay, uniform-price and discriminatory-price auctions, respectively. Suppose there exists a symmetric increasing equilibrium in the game of first-stage bidding under scheme $\mathcal{F}$, in which every bidder bids according to a symmetric increasing bid function $\varphi^{\mathcal{F}} : [0, \bar{x}] \to \mathbb{R}$. Now consider its associated direct revelation mechanism (DRM) where bidders are required to report their first-stage types ($X_i$'s). Given a report profile $x = (x_i, x_{-i})$, a DRM is described by an entry right assignment rule $\{y_i(x)\}_{i=1}^N$ and an entry fee payment rule $\{p_i(x)\}_{i=1}^N$. The assignment rule $y_i(x_i, x_{-i}) = 1$ if $x_i > X_{n;-i}$ (bidder $i$ gains entry), and $y_i(x_i, x_{-i}) = 0$ if $x_i < X_{n;-i}$ (bidder $i$ is excluded from entry).[18] Since $n$ bidders will be selected based on their reports, $\sum_{i=1}^N y_i(x) = n$. Let $m_i^{\mathcal{F}}(x_i)$ denote bidder $i$'s expected entry fee payment when she reports $x_i$ and all the other bidders report their types ($X_{-i}$) truthfully. Then $m_i^{\mathcal{F}}(x_i) = E[p_i(x_i, X_{-i})]$. According to the specific entry fee payment rule, $m_i^I(x_i) = 0$, $m_i^A(x_i) = \varphi^A(x_i)$, $m_i^U(x_i) = E[\varphi^U(X_{n;-i})\mathbf{1}_{x_i > X_{n;-i}}] = \int_0^{x_i} \varphi^U(t)\,\mathrm{d}F_{X_{n;-i}}(t)$, and $m_i^D(x_i) = E[\varphi^D(x_i)\mathbf{1}_{x_i > X_{n;-i}}] = \varphi^D(x_i)F_{X_{n;-i}}(x_i)$.[19]

17 Note that the summary statistic ranking may create inefficient allocations, since a bidder with the highest private value component may not end up with the highest summary statistic to win the asset. However, with efficient entry lower bidders are excluded from the final auction and this would reduce the magnitude of the inefficiency.

18 When $x_i = X_{n;-i}$, tie will be resolved at random. Note that a tie occurs with probability zero in any symmetric increasing equilibrium.

19 Throughout we use $\mathbf{1}_B$ to denote the indicator function, taking value 1 if the event $B$ is true and 0 if $B$ is not true.

Suppose all but bidder $i$ report their types ($X_{-i}$) truthfully. Then if bidder $i$ reports $x_i'$, she will make an entry fee payment $p_i(x_i', X_{-i})$. In addition, if $x_i' > X_{n;-i}$, she will be selected for the second stage, in which case she will incur entry cost $c$, receive an entry award $K$ (the net effect is to incur an adjusted entry cost $\hat{c} = c - K$), and engage in the final auction. By the payoff equivalence result (Proposition 1), bidder $i$'s expected profit from attending the second-stage auction is $E\mathbf{1}_{S_i > \widehat{S}_{1;-i}}(S_i - \widehat{S}_{1;-i})$. Thus, when bidder $i$ reports $x_i'$ while all her competitors report their types truthfully, her expected profit is:

$$\begin{aligned}
\pi_i(x_i', x_i) &= E\mathbf{1}_{X_{n;-i} < x_i'}\left[\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\left(S_i - \widehat{S}_{1;-i}\right) - \hat{c}\right] - m_i^{\mathcal{F}}\left(x_i'\right) \\
&= E_{X_{n;-i} < x_i'} E\left(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\left(S_i - \widehat{S}_{1;-i}\right) \mid X_{n;-i}\right) - \hat{c} F_{X_{n;-i}}\left(x_i'\right) - m_i^{\mathcal{F}}\left(x_i'\right) \\
&= \int_0^{x_i'} E\left(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\left(S_i - \widehat{S}_{1;-i}\right) \mid X_{n;-i} = t\right) \mathrm{d}F_{X_{n;-i}}(t) - \hat{c} F_{X_{n;-i}}\left(x_i'\right) \\
&\quad - m_i^{\mathcal{F}}\left(x_i'\right).
\end{aligned} \tag{3}$$

Incentive compatibility (IC) implies the following first order condition (FOC):

$$\begin{aligned}
\left.\frac{\partial \pi_i}{\partial x_i'}\right|_{x_i' = x_i} &= \underbrace{E\left(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\left(S_i - \widehat{S}_{1;-i}\right) \mid X_{n;-i} = x_i\right)}_{E\Omega(x_i)} f_{X_{n;-i}}(x_i) - \hat{c} f_{X_{n;-i}}(x_i) - \frac{\mathrm{d}m_i^{\mathcal{F}}(x_i)}{\mathrm{d}x_i} \\
&= E\Omega(x_i) f_{X_{n;-i}}(x_i) - \hat{c} f_{X_{n;-i}}(x_i) - \frac{\mathrm{d}m_i^{\mathcal{F}}(x_i)}{\mathrm{d}x_i} \\
&= 0.
\end{aligned} \tag{4}$$

The above FOC is a necessary condition implied from a symmetric increasing equilibrium in the original game of first-stage bidding. Based on this condition we can first analyze the current design of indicative bidding.

### *5.1. Non-binding indicative bidding*

**Proposition 2.** *Under the current design of indicative bidding, no symmetric increasing equilibrium exists* (*generically*).

**Proof.** Under the current design of indicative bidding no entry award is provided, so $\hat{c} = c$; no fee needs to be paid for entry, so $m_i^I = 0$. Hence Eq. (4) becomes $E\Omega(x_i) = c$. But this condition only holds at points with probability measure zero. (Rigorously speaking, it might hold for some extremely special distribution, but that, if any, would not be a generic case.) Therefore by the Revelation Principle, in the original game of indicative bidding, no symmetric increasing equilibrium exists (generically). □

Given the widespread use of indicative bidding and the tremendous amount of money involved in practice, the above nonexistence result is remarkable.

This nonexistence result can be understood from FOC (4). When all bidders but $i$ report their types truthfully, slightly deviating from the report $x_i$ will only have marginal effect for bidder $i$

when she is the marginal entrant (i.e., when $x_i = X_{n;-i}$).[20] Conditional on being the marginal entrant, bidder $i$'s marginal expected gain from attending the second-stage auction is $E\Omega(x_i)$ while the marginal cost from entry is the entry cost $c$. When the marginal gain is higher, bidder $i$ would have an incentive to over-report her type; when the marginal gain is lower, she would have an incentive to under-report. The balance only occurs at points with probability measure zero (generically), therefore a symmetric increasing equilibrium does not exist (generically).

The intuition is simpler in the pure entry case. When entry does not involve value updating, $E\Omega(x_i) = 0$ since being the marginal entrant one can never become the final winner in the second stage. Therefore, given that everyone else is bidding according to some symmetric increasing function, bidder $i$ would have the incentive to underbid.[21]

FOC (4) also suggests the following result regarding entry subsidization policy under the current design of indicative bidding:

**Corollary 1.** *In the pure entry case, if the entry cost is fully subsidized by the auctioneer* (*i.e.,* $K = c$)*, then any strictly increasing bid function can arise in a symmetric equilibrium; but in the case of non-trivial value updating, no level of entry subsidy* (*partial or full, positive or negative*) *can restore a symmetric increasing equilibrium* (*generically*).

In the pure entry case, the entry cost is the only source for the nonexistence result, so simply subsidizing the entry cost restores the IC condition on the bidders' side (but it results in multiple equilibria, since any increasing bid function will produce a symmetric equilibrium).[22] In more general cases, the non-trivial value updating and entry cost are both sources for the nonexistence result. Given non-trivial value updating, the expected profit for the marginal entrant varies with type, thus a pre-specified pure entry subsidy cannot restore the IC condition.

Given the above nonexistence result regarding the current design of indicative bidding, we now turn our attention to alternative schemes with binding first-stage bidding.

### *5.2. Binding first-stage bidding*

Under the alternative schemes, the first-stage bids are binding; they will now indicate the bidders' firm commitment to pay for entry. Under such schemes bidders first bid for entry and then, upon being selected, bid for the asset in the second stage.

Analogously to the analysis of the second-stage auction, we assume that the alternative schemes conducted in the first stage are also standard auctions, which are defined as follows.

**Definition 3** *(Standard First-Stage Auctions).* Auctions in which entry rights are awarded to the bidders with the $n$ highest bids, and the bidder with the lowest possible type pays zero expected entry fee.

---

[20] When $x_i > X_{n;-i}$, bidder $i$ will be selected if she reports truthfully, and when she under-reports by a sufficiently small amount, she will still be selected; when $x_i < X_{n;-i}$, $i$ will not be selected if reporting truthfully, and when she over-reports by a sufficiently small amount, she will still not be selected.

[21] Also see Milgrom (2004, pp. 229–230) for an alternative intuition for the nonexistence result in this pure entry case.

[22] Note that this result holds when $n \geqslant 2$, which is assumed in this section. When $n = 1$, IC condition does not hold since anyone will submit the highest bid trying to become the only entrant.

The all-pay auction, uniform-price auction, and discriminatory-price auction are all standard auctions. We begin by showing that under the alternative schemes the expected entry fee must be the same in equilibrium.

**Lemma 2.** *Given an entry award $K$, in any symmetric increasing equilibrium, the expected entry fee is the same under all (standard) auction formats. Hence,*

$$\varphi^A(x) = \int_0^x \varphi^U(t)\,\mathrm{d}F_{X_{n;-i}}(t) = \varphi^D(x) F_{X_{n;-i}}(x). \tag{5}$$

**Proof.** The entry fee equivalence follows from Eq. (4) and the boundary condition $m_i^{\mathcal{F}}(0) = 0$ (by the definition of standard auctions). Equation (5) then follows from the payment rules of the three alternative entry auctions. □

The following result identifies an exact condition under which a customary auction (a uniform-price auction or a discriminatory-price auction) can induce efficient entry.

**Proposition 3.** *Under a customary auction, there exists a unique symmetric increasing equilibrium if $E\Omega(x)$ is strictly increasing in $x \in [0, \bar{x}]$.*[23]

**Proof.** See Appendix A. □

$E\Omega(x_i) = E(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}(S_i - \widehat{S}_{1;-i}) \mid X_{n;-i} = x_i)$ is bidder $i$'s expected gain from the second-stage auction conditional on being the marginal entrant (i.e., $X_{n;-i} = x_i$). The proof in Appendix A shows that if this term is strictly increasing in $x_i$, then under a uniform-price auction, the unique symmetric increasing equilibrium bid function is as follows:

$$\varphi^U(x) = E\Omega(x) - \hat{c}. \tag{6}$$

In equilibrium each bidder bids as if she is the "marginal bidder," i.e., each bidder bids an amount equal to the net expected gain from entry assuming that she is the marginal entrant.

In the pure entry case, $E\Omega(x_i) = 0$. Therefore no symmetric increasing equilibrium exists under a uniform-price auction. It can be easily verified that under a discriminatory-price auction, no symmetric increasing equilibrium exists either.

To gain some insight about the condition under which $E\Omega(\cdot)$ is strictly increasing, we consider a simplified situation as follows: $X \in [0, 1]$. $Y$ is another private value component and has a Bernoulli distribution taking value 1 with probability $p$, and 0 with probability $1 - p$. (A simple interpretation is that the second-stage signal takes value either "High" or "Low.") A sufficient condition is given by the following result:

**Proposition 4.** *In the preceding example, a unique symmetric increasing equilibrium exists under customary auctions if $X$'s hazard rate function $H(\cdot)$ is strictly increasing.*

**Proof.** See Appendix A. □

[23] For the uniform-price auction, the statement is stronger: a unique symmetric increasing equilibrium exists if and only if $E\Omega(\cdot)$ is strictly increasing. This can be seen from the proof of Proposition 3.

Note that the monotonicity of $H(x)$ is implied by the log-concavity of $f(x)$.[24] Thus in this example, the condition required to induce efficient entry under customary auctions is not stringent.

In general, the monotonicity of $E\Omega$ imposes restrictions on the heterogeneity (the distribution) of the bidders, hence efficient entry is not always guaranteed under customary auctions. However, this restriction is completely removed when we consider an all-pay auction.

**Proposition 5.** *Under an all-pay auction, the auctioneer can always select an entry award $K$, if necessary, to induce a unique symmetric increasing equilibrium.*

**Proof.** See Appendix A. □

This result is consistent with the major finding in Fullerton and McAfee. In the context of a research tournament, they showed that an all-pay auction with an entry award always induces efficient entry, regardless of the heterogeneity of contestants.

A further question is how large $K$ needs to be in order to induce efficient entry. Note that

$$\frac{\mathrm{d}\varphi^A(x)}{\mathrm{d}x} = \varphi^U(x) f_{X_{n;-i}}(x) = \big(E\Omega(x) - \hat{c}\big) f_{X_{n;-i}}(x).$$

Letting $\underline{\Omega} = \min_{x\in[0,\bar{x}]} E\Omega(x)$, then $\varphi^A(x)$ is strictly increasing if and only if $K > c - \underline{\Omega}$. In the pure entry case, $E\Omega(x) = 0$, so $K$ should be strictly greater than $c$ for an all-pay auction to induce a symmetric increasing equilibrium; when entry involves non-trivial value updating, $\underline{\Omega} > 0$ since $\mathrm{Var}(Y) > 0$, thus setting $K = c$ is large enough to induce a symmetric increasing equilibrium.

When the entry cost $c$ is relatively small, such that $c < \underline{\Omega}$, no entry award is needed for an all-pay auction to work; however, when the entry cost is substantial, in particular when $c > \underline{\Omega}$, then a positive entry award will have to be used.

Note that in the symmetric increasing equilibrium bids under an all-pay auction are always positive. But bids under a customary auction can be negative, in which case the bids are meant as demands for entry subsidies.

### *5.3. Payoff equivalence*

When all alternative schemes work to induce efficient entry, which specific scheme should the auctioneer choose? It turns out that the auctioneer will be indifferent, due to the following payoff-equivalence result:

**Proposition 6.** *Under the alternative schemes with binding first-round bidding, in the symmetric increasing equilibrium all* (*standard*) *auctions are equivalent in terms of the expected profit to the bidders and the expected revenue to the seller.*

**Proof.** Lemma 2 establishes the entry fee equivalence. Plugging $m_i^{\mathcal{F}}(x_i)$ into the value function $\pi_i(x_i) =: \pi_i(x_i, x_i)$ in (3), we have bidders' profit equivalence. Since in the symmetric increasing equilibrium all auctions (eventually) lead to the same winner of the asset, the seller's revenue equivalence also follows. □

[24] A proof can be found in Goeree and Offerman (2003).

Combining the payoff equivalence in the second-stage auction (Proposition 1) and the payoff equivalence in the first-stage auction (Proposition 6), we have actually shown that any combination of two-stage (standard) auctions generates the same expected payoffs in the symmetric increasing equilibrium. Thus a revenue equivalence result is extended to this two-stage auction environment.

Since the entry award $K$ is not needed in customary auctions, Proposition 6 implies the following corollary:

**Corollary 2.** *While $K$ facilitates the efficient entry under an all-pay auction, it does not affect the equilibrium expected payoffs.*

### *5.4. Optimal number of final bidders*

In all the previous sections we analyzed the game conditional on the assumption that $n$ is given (and $n \geqslant 2$). In this section we relax this assumption and explore how the optimal number of final bidders ($n^*$) is determined.[25]

First let us consider the case $n = 1$. Given the assumption that the auctioneer is unable to set a reserve price, $n = 1$ implies that only the highest bidder in the first-stage is selected and the asset is awarded to this bidder with price 0. In this case, all the expected revenue comes from the first-stage entry fee payments and the two-stage bidding is essentially reduced to a single-stage auction.

We again consider an IC direct revelation mechanism in the first stage. Assuming that all rivals report their types truthfully, bidder $i$'s expected profit by reporting $x_i'$ given her true type $x_i$ is:

$$\pi_i(x_i', x_i) = (x_i + EY)F_{X_{1;-i}}(x_i') - \hat{c}F_{X_{1;-i}}(x_i') - m_i^{\mathcal{F}}(x_i')$$

and (IC) implies

$$\begin{aligned}\left.\frac{\partial \pi_i}{\partial x_i'}\right|_{x_i'=x_i} &= (x_i + EY - \hat{c})f_{X_{1;-i}}(x_i) - \frac{\mathrm{d}m_i^{\mathcal{F}}(x_i)}{\mathrm{d}x_i} \\ &= 0.\end{aligned}$$

Based on this FOC we can verify that a unique symmetric increasing equilibrium exists under both customary auctions (in particular, $\varphi^U(x) = x + EY - \hat{c}$). A unique symmetric increasing equilibrium also exists under an all-pay auction with large enough $K$. In equilibrium, it can be verified that the total expected rent to the bidders is

$$E\Pi = N\int_0^{\bar{x}} \bigl(1 - F(x)\bigr)F^{N-1}(x)\,\mathrm{d}x$$

and the expected revenue is

$$ER = EX_{2;N} + EY - c. \tag{7}$$

Intuitively, when only one bidder will be selected, the bidders will submit bids in the first stage assuming that there is a common value component ($EY$) in the second stage.

[25] More accurately, the optimal number of bidders analyzed in this section is the expected revenue maximizing number of bidders assuming that the seller is unable to set a reserve price.

Now we consider the case $n \geqslant 2$, where competitive bidding is maintained in the second stage.

Let $e(x_i)$ denote bidder $i$'s conditional expected entry fee payment net of entry cost and entry award given a first-stage signal $x_i$. Then

$$e(x_i) = \int_0^{x_i} E\Omega(x)\,\mathrm{d}F_{X_{n;-i}}(x).$$

The bidders' expected total "net" payment for the entry can then be computed as follows:

$$\begin{aligned} EM &= NEe(X_i) \\ &= N\int_0^{\bar{x}}\int_0^{x_i} E\Omega(x)\,\mathrm{d}F_{X_{n;-i}}(x)\,\mathrm{d}F(x_i) \\ &= N\int_0^{\bar{x}}\bigl(1-F(x)\bigr)E\Omega(x)\,\mathrm{d}F_{X_{n;-i}}(x). \end{aligned} \tag{8}$$

It can be shown (in the proof of Proposition 7 below) that in the private value updating case, the expected revenue is

$$ER = E\widehat{S}_{2;n} + EM - nc \tag{9}$$

where $EM$ is given by (8) with $\lambda = 1$.

Similarly, in the common value updating case, the expected revenue is

$$ER = E\widehat{S}_{2;n} + \frac{N-1}{N}EY + EM - nc \tag{10}$$

where $EM$ is given by (8) with $\lambda = \frac{1}{N}$.

So in all the cases, the expected revenue can be decomposed into the sum of the two stages' payments (the payment for the asset and the net payment for entry), less the total entry cost.

**Proposition 7.** *Let $n^*$ be the optimal number of bidders to be selected for the second-stage auction. In the private value updating case, $n^*$ maximizes the expected revenue given in* (7) *and* (9). *In the common value updating case, $n^*$ maximizes the expected revenue given in* (7) *and* (10).

**Proof.** See Appendix A. □

In the pure entry case, since $EY = 0$ and $EM = 0$, the optimal number of bidders is $n^* = 1$. This result is intuitive: when the second-stage learning does not have value it is optimal to completely eliminate the competition in the second-stage bidding. When entry involves non-trivial value updating, however, as we will see in the examples given in the next section, $n^*$ can be two or higher depending on the specific contexts. Thus regarding the optimal number of entrants, the implication of our model is quite different from that of Fullerton and McAfee, who show that $n^*$ is always two.

By inspecting Eqs. (7), (9) and (10), we also obtain the following result:

**Proposition 8.** *In any symmetric increasing equilibrium, the seller bears all the entry cost indirectly.*

Other things being equal, increasing the total entry cost ($nc$) by \$1, the expected revenue to the seller is reduced by \$1 accordingly. In equilibrium, the entry cost does not affect the bidder's expected profit; all the entry cost is borne by the seller indirectly. This result has been obtained in the earlier literature when bidders only acquire private information after entry occurs (see, e.g., French and McCormick, 1984; McAfee and McMillan, 1987; Levin and Smith, 1994). But we show this under our more general setup in which bidders acquire private information both before and after entry occurs. It can thus be regarded as a robust result in the literature of auctions with costly entry. The implication is that although the bidders incur the entry costs directly, it is the seller, not the bidders, who should be concerned about the effect of the entry cost on the outcome of the sales. This result also explains why optimal auctions with costly entry are typically characterized by a limited number of final bidders.

## 6. Solved examples

The complicated functional forms involving revenue maximization generally prevent us from obtaining analytical solutions for an optimal number of final bidders. We thus provide examples in this section, the solving of which was aided by using Monte Carlo simulation methods.

Consider again the Bernoulli example mentioned in Section 5.2, where the second-stage private signal is either "High" ($Y = 1$) with probability $p$ or "Low" ($Y = 0$) with probability $(1 - p)$. To simplify the computation we assume that $X$ is distributed as Uniform[0, 1]. We will consider the private value updating and the common value updating cases in order.

### 6.1. The private value updating case

In this case the total value is $X + Y$. Since a Uniform random variable has a strictly increasing hazard rate, by Proposition 4 a unique symmetric increasing equilibrium exists under both customary auctions. (A unique symmetric increasing equilibrium also exists under an all-pay auction with high enough $K$ by Proposition 5.) It can be verified that the expected profit from attending the second-stage auction conditional on being the marginal entrant with a first-stage value $x_i$ is[26]

$$E\Omega(x_i) = p(1-p)^{n-1}\left(\frac{1}{n} + \frac{n-1}{n}x_i\right).$$

The equilibrium first-stage bid function under a uniform-price auction thus takes the following simple (linear) functional form:

$$\varphi^U(x_i) = p(1-p)^{n-1}\left(\frac{1}{n} + \frac{n-1}{n}x_i\right) - \hat{c}.$$

To get some sense about how the optimal number of entrant firms ($n^*$) changes with the underlying parameters, we perform four sets of simulations: (1) $N = 100$, $p = 0.5$; (2) $N = 100$, $p = 0.8$; (3) $N = 50$, $p = 0.5$; (4) $N = 50$, $p = 0.8$. In each set, the optimal $n$ is computed according to a different level of entry costs. In terms of the percentage of the total maximal value

[26] The detailed derivations in this and next subsections are analogous to that in the proof of Proposition 4. A nice feature of this example is that it is very clear when a marginal entrant wins the object. Since the marginal entrant has the lowest first-stage signal, she wins if and only if she gets a good shock in the second stage but all the other entrants get bad shocks, which occurs with probability $p(1-p)^{n-1}$.

(2 in this case), the following levels of entry cost are considered: 0.01, 0.05, 0.1, 0.2, 0.5, 1%. The results are reported in Table 1 in Appendix A. (In particular, the values of $E\widehat{S}_{2;n}$ are obtained from simulations.)

Several observations are worth noting. First, other things being equal, $n^*$ is non-increasing in entry cost $c$, which is intuitive and predicted by monotone comparative statistics. Second, the optimal number of bidders for maximizing expected revenue is generally finite and small. For example, in the case $p = 0.5$, the optimal number of entrants ranges from 4 to 11. Third, other things being equal, the effect of $N$ on $n^*$ is ambiguous. When $p = 0.8$, $n^*$ is consistently higher when $N = 100$ than when $N = 50$; while when $p = 0.5$, the direction becomes ambiguous. Finally, in this example, the total "net" entry fee payment defined in Eq. (8), $EM$, is unambiguously increasing in $N$: the increased competition among first-stage bidders drives up the total net entry fee. In fact, $EM$ can be solved analytically in this case:

$$EM = p(1-p)^{n-1}\left[1 + \frac{(n-1)(N-n)}{N+1}\right],$$

which is increasing in $N$.

### 6.2. The common value updating case

Now consider the common value updating case, where the asset's value for entrant $i$ is $X_i + \frac{1}{N}\sum_{j=1}^{N} Y_j$. It can be shown that a bidder's expected profit from attending a second-stage auction conditional on being a marginal entrant with first-stage signal $x_i$ is:

$$E\Omega(x_i) = \begin{cases} p(1-p)^{n-1}\frac{1}{n}(\frac{1}{N})^n(\frac{1}{1-x_i})^{n-1} & \text{if } x_i < 1 - \frac{1}{N}, \\ p(1-p)^{n-1}\frac{1}{nN} & \text{if } x_i = 1 - \frac{1}{N}, \\ p(1-p)^{n-1}[\frac{1}{N} - (1-\frac{1}{n})(1-x_i)] & \text{if } x_i > 1 - \frac{1}{N}. \end{cases}$$

Again, $E\Omega(\cdot)$ is strictly increasing, hence both customary auctions can induce efficient entry in this common value updating case.

Analogously to the private value updating case, we produce Table 2 in Appendix A. As we can see immediately, now the optimal number of bidders is two in all the cases.

In the common value updating case, $n$ does not affect the expected total value generated from the sale by much. But it matters for rent extraction. When $n = 1$, the winner earns all the private information rent since we assume that the seller is unable to set a reserve price (see Eq. (7)). However, when $n \geqslant 2$, the competitive bidding helps the seller in rent extraction (since $EM > 0$). Specifically, the seller extracts some of the bidders' rent by combining the first-stage entry fee payment scheme and the second-stage competitive bidding. This explains why $n = 1$ is not optimal.

That $n > 2$ is not optimal in all the cases is mainly due to our common value formulation. Under our formulation, a substantial portion of the common value component in the term, $\frac{N-1}{N}EY$, does not give rise to the private information rent of the bidders (only the summary statistic, $X + \frac{1}{N}Y$, matters for the private rent). Therefore, it does not pay for the seller to set $n > 2$, since there is little private rent left to extract beyond the point $n = 2$, yet there is a real cost in admitting one more entrant. This explains why $n > 2$ is not optimal in all the cases considered in this example.

Finally, note that the expected net entry fee payment $EM$ is now decreasing in $N$, which is in contrast with the private value updating case above. We know that in traditional common value auctions expected revenue may decrease as competition increases. Here we have another similar finding, that is, the net entry fee payment may decrease as the competition for entry rights increases. We conjecture that this finding is related to the winner's curse effect specific to the common value setting.

## 7. Discussion

In this section, we discuss some limitations of our analysis and the robustness of our results. We also address some issues not covered in our previous analysis.

First, we assume that the entry cost is the same for all entrants. The justification for this assumption is the specific institutional setup we are trying to model. In our motivating example of the electrical generating asset sales, after bidders are selected, they will be permitted comprehensive access to the generating facilities to conduct due diligence intensively. During this process, they will be allowed to meet senior management and personnel, study equipment conditions and operating history, and evaluate supply contracts and employment agreements, etc. This process is strictly monitored by the investment banker, usually serving as the financial adviser to the selling firm. Since the due diligence procedure is highly controlled and closely monitored, we believe that it is reasonable to assume that the entry cost for each entrant bidder is more or less the same. A more general model should allow for different entry costs for different bidders. However, if we do allow for this generalization, the analysis can easily become intractable. In fact, no symmetric increasing equilibrium may exist even under the alternative schemes with binding first-round bidding. This can be seen from Eq. (4): the IC condition is unlikely to be satisfied simultaneously for all bidders with different $c_i$'s. Therefore, assuming the same entry cost for each entrant is probably a necessary first step in building a model to analyze indicative bidding.

Second, an implicit assumption in our analysis is that whenever selected, bidders cannot withdraw from the auction process and must incur the entry costs. Our justification is that without going through the due diligence process, no additional information can be learned about the asset. (Again, note that the information aggregation process is closely monitored by the auctioneer in practice.) Therefore, it does not help for a bidder to enter the final phase only to withdraw without going through the due diligence process. In our model, we show that in equilibrium, the seller bears all the entry cost indirectly, and the bidders will end up with positive expected profit conditional on being selected if they follow the equilibrium play. This further suggests that the bidder would not have strict incentive to withdraw from the auction after being selected.

Third, in this paper we assume that the total value function takes an additive form and the signals from the two stages ($X$ and $Y$) are independent with nonnegative supports. To see that these assumptions are not crucial for our main results to hold, we consider an alternative formulation as follows. In the first stage, each potential bidder is endowed with a first-stage signal, or a first-stage "type" $X_i$, which is private information known only to the bidder. Ex ante, $X$ is distributed as $F(\cdot)$ with support $[\underline{x}, \bar{x}]$. After going through the due diligence process, the bidder will learn a signal $Y_i$. The conditional distribution of $Y_i$ given $X_i$ is $G(\cdot|X_i)$. While $X_i$'s are independent of each other, each $X_i$ and $Y_i$ are affiliated, implying that $G(y|x)$ is nonincreasing in $x$. To insure that signals are informative, we further assume that whenever $x' > x$, the distribution $G(\cdot|x')$ strictly dominates $G(\cdot|x)$. The value of the asset to bidder $i$ given the vector $Y = (Y_1, Y_2, \dots, Y_N)$ is given by $V_i = u_i(Y_1, Y_2, \dots, Y_N) = u(Y_i, Y_{-i})$ for some function $u$ on $R^N$. That is, each bidder's valuation is a symmetric function of the other signals. We also assume that $u$ is nonnegative, continuous, and increasing in all its arguments.[27] Under this information structure and valuation setup, we again look for the equilibrium with efficient entry. Given efficient entry we can compute the expected profit for a bidder (with type $x_i$) from attending the second-stage auction conditional on being the marginal entrant. As before, denote this term as $E\Omega(x_i)$. In the associated direct game, if everyone else report their types truthfully, a report by bidder $i$ with a sufficiently small deviation from her true type will only have marginal effect on the outcome when her type $x_i$ is tied with the marginal entrant's type (i.e., $X_{n;-i} = x_i$). So when the first-stage bidding is nonbinding, incentive compatibility implies

$$\left[E\Omega(x_i) - c\right] \cdot f_{X_{n;-i}}(x_i) = 0.$$

Note that this condition is exactly the same as derived before. Thus the non-existence result regarding the current design of indicative bidding remains valid (generically). Under alternative schemes with binding first-round bidding, it can be easily seen that condition (4) continues to hold, which implies that the existence results regarding alternative schemes remain valid as well.

This example demonstrates that the key in our analysis is the term $E\Omega(x_i)$. Once we have derived $E\Omega(x_i)$ from any given signal structure and value formulation, our analysis can be carried out exactly the same as before in terms of $E\Omega(x_i)$. Given any generic functional form of $E\Omega$, we know that the IC condition is unlikely to be "balanced" under the current design of indicative bidding, while it can be "balanced" under the alternative schemes under certain conditions. Therefore, for our main results to hold, the specific value formulation, the independence between $X$ and $Y$, and the nonnegative support of $Y$ can all be dispensed with.

Fourth, we assume that the seller is unable to set a reserve price. This assumption imposes restrictions on the characterization of optimal auctions, but we believe that the other results in this paper can still carry over when a reserve price is incorporated in the model. Note that in particular, the argument for the nonexistence result does not rely on whether or not there is a reserve price. Though optimal auctions characterized by both an optimal number of bidders and an optimal reserve price remain to be explored for future research, we conjecture that the seller does not lose much when he is unable to set a reserve price. Two findings may support this conjecture. First, when bidders only learn valuations after entry, the optimal reserve price is zero whenever charging an entry fee is feasible. This result holds under both private value and common value auctions.[28] Second, when bidders know their valuations before entry, Ye (2001) shows that for sufficiently large $N$, any auction with a reserve price is revenue-dominated by an optimal two-stage auction not involving reserve prices. These results suggest that the reserve price may play little, if any, role in optimal auctions with costly entry.

Fifth, we show that no symmetric increasing equilibrium exists under the current indicative bidding mechanism, but we have not addressed whether there exist other equilibria. That issue is left for future research.[29] There is also the question of the importance of symmetric increasing

[27] This formulation encompasses the private value case $V_i = Y_i$, or a mixture of private value and common value case, e.g., $V_i = \alpha Y_i + \beta\xi(Y_{-i})$, where $\alpha, \beta > 0$ and $\xi$ is nondecreasing in each of its arguments.

[28] See, for example, French and McCormick (1984), McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), and Levin and Smith (1994). The intuition goes as follows. The endogenous entry process will drive down bidders' expected rent to zero (or approximately zero due to the integer constraint of the number of entrant bidders). Therefore in equilibrium the expected revenue is equal to or approximately equal to the expected total surplus. To maximize the total surplus, the reserve price should be set at the same level as the seller's own valuation. Ye (2004) further extends this result to the setting where bidders' beliefs about one another's valuation can be updated through entry.

[29] One attempt would be to look for an equilibrium with endogenous sharing rules (or tie-breaking rule), as originally proposed by Simon and Zame (1990) and further extended by Jackson et al. (2002) to games with incomplete information.

equilibrium. The auction literature focuses on symmetric increasing equilibria, either for technical convenience or because people believe that playing symmetric equilibria would be more "focal" than playing other equilibria. However, how bidders actually bid in the real world can be a completely different story. As is well documented in the experimental auction literature, subjects may even fail to follow dominant strategies (Kagel, 1995). But here, we employ symmetric increasing equilibrium due to the requirement for efficient entry, which is central to our current research.

Finally, in the previous section, we did not address the issue regarding the benefit of running indicative bidding. Why would the seller pre-select a certain number of bidders through indicative bidding? Why don't sellers just let bidders make their own entry decisions and run a single-stage auction instead? One simple answer is that there might be some constraint on how many bidders can be admitted for the due diligence process. The seller may need to consider things like how many data rooms can be provided (capacity constraint), or how to prevent the information leakage about the power plants (security concern), etc. These constraints on the seller's side are not modeled in this paper, but they may discourage sellers from admitting too many bidders to the final stage. Indicative bidding solves this problem by selecting a targeted number of bidders for the final auction. An alternative answer has to do with the difference in entry processes. The entry process induced by indicative bidding is essentially a deterministic entry process, in the sense that a deterministic number of bidders enter the auction. If indicative bidding is not employed, bidders may simultaneously make their entry decisions based on their first-stage private signals about the sale. Following our general model, we can show that there exists a cut-off equilibrium in which each bidder enters the auction if and only if her first-stage private signal ($x_i$) is higher than some cut-off point ($x^*$). In equilibrium, the number of bidders who actually choose to enter the auction is a random variable.[30] So participation is a stochastic process, which may impose "coordination" costs on the seller's side (Levin and Smith, 1994): when the realization of the actual number of final bidders $n$ is low (e.g., $n = 1$ or even worse, $n = 0$), the seller may suffer from the low revenue generated. This consideration would favor a deterministic entry process, like the one induced by indicative bidding.

## 8. Conclusion

To our knowledge, this research represents a first attempt to address the issue of indicative bidding, a widely used practice in auctions of highly valuable and complex objects. Given the complexity of high value asset sales, our model neglects some institutional details in order to capture the essence of indicative bidding: First, there is a preliminary stage and the bidders are asked to submit nonbinding bids; second, the auctioneer restrict the number of participants, and the selection is mainly based on the prices indicated in the preliminary bids; third, there is a substantial entry cost for each final bidder to acquire information and prepare a final bid, and finally, information updating can be quite substantial during the due diligence stage. In all the cases we have analyzed, we show that no symmetric increasing equilibrium exists under the current design of indicative bidding. Since there is no guarantee that the most qualified bidders will be reliably selected, the efficiency loss could be potentially substantial.

---

However their results only apply to games where individual rationality condition is automatically satisfied, which is typically a problem in our case due to the costly entry. So their sufficient conditions cannot be applied to check the existence of equilibrium in our case.

[30] It is distributed as Binomial $(N, 1 - F(x^*))$.

Our previous section suggests that the nonexistence result regarding the current design of indicative bidding is fairly robust. Actually even if a symmetric increasing equilibrium exists under some alternative setting, a related concern would be the huge potential multiplicity of equilibria. Suppose that some symmetric increasing equilibrium bidding function $\varphi$ exists, then for any (positive) monotone transformation $\phi$, the bidding function $\phi \circ \varphi$ is also a symmetric increasing equilibrium. In the face of such vast multiplicity, coordination could be extremely difficult.[31]

One caveat in interpreting the nonexistence result is that the absence of a symmetric increasing equilibrium does not necessarily lead to inefficient entry—it merely implies that efficient entry is not guaranteed from game theoretical point of view. Even in the situations where no equilibrium exists, there remain both theoretical and practical issues about how the bidders would actually perform. This suggests the need for complementary approaches, such as empirical or experimental approaches in the future analysis of indicative bidding. The practicability of one selling mechanism depends not only on its underlying equilibrium properties (e.g., the existence or uniqueness of some desirable equilibrium), but also on a variety of other considerations. Thus the continuing widespread use of the indicative bidding may be an indication that despite its potential efficiency drawback, it may exhibit offsetting advantages in other dimensions such as the simplicity and transparency of the selling rules and the relative ease of implementation. Implementation of the improved mechanisms proposed here may face several obstacles. First, the industry may not accept the idea of paying entry fees whether or not they win the object even if paying for entry could be the only solution capable of inducing efficient entry. Second, industry bidders face difficulties in determining equilibrium strategies under the alternative schemes (though this concern can be alleviated if firms hire professional and sophisticated consultants to bid, as observed in the FCC spectrum auctions).[32]

On the other hand, the widespread use of one mechanism or institution does not necessarily mean that such a mechanism or institution is already operating in full efficiency and cannot be improved. It may simply reflect the fact that better mechanisms have not been identified, or the conditions for improvements are not yet ready. This paper's demonstration of the (theoretical) advantage of alternative mechanisms in inducing efficient entry lays the groundwork for future experimentation to determine whether improvements are worthwhile. The popularity of indicative bidding suggests a large potential payoff for adoption of improved alternatives.

Despite the nonexistence result, this research does justify the general practice of running two-stage auctions in the costly entry environment. As we have shown, in equilibrium all the entry cost is indirectly borne by the seller; hence it is to the seller's best interest to limit the participation. In a more general setting, Barzel (1982) touches on the issue of measurement cost and the market response to transactions with pre-sale measurement costs. He points out that since the resource expenditures on pre-sale measurement are wasteful, the exchange parties will form contracts, organizations, or activities to reduce the sunk cost.[33] The current two-stage auction

[31] One may suggest the possibility for the existence of a symmetric increasing equilibrium based on reputation considerations. That is, due to relational or reputational concern, bidders may bid "truthfully" in the indicative bidding stage so that a symmetric increasing equilibrium can be induced. However, such reputation effect, if any, would be greatly reduced in our context. First, it is unlikely that a bidder can enter auctions with billion-dollar assets repeatedly. Second, since the information updating after entry is usually substantial, detecting a "cheating" in indicative bidding would be extremely difficult, if not entirely impossible.

[32] The game with binding first-round bidding can be quite complicated, as the equilibrium bidding involves sophistications such as the computation of some nontrivial conditional expectations.

[33] A related empirical analysis of this can be found in Leffler et al. (2000).

mechanism can thus be regarded as a market response to reduce the entry cost and increase expected revenue.

Compared to earlier literature, this research not only justifies the optimality of restricting participation in the costly entry environment, but also addresses the issue of how to select the most qualified bidders for the final sale. Motivated by the analysis of indicative bidding, this research ends with a theory of two-stage auctions. The framework developed in this research is quite general, which allows us to evaluate the current indicative bidding mechanism, study alternative selection schemes, and also explore general features regarding auctions with costly entry. Given the widespread use of indicative bidding and the tremendous amount of money involved, we hope that our results can enhance our understanding of two-stage auctions in general and indicative bidding in particular.

## Acknowledgments

This paper is a revised chapter of my PhD thesis at Stanford University. I am indebted to Paul Milgrom, Robert Wilson, John McMillan, and Patrick Bajari for their generous advice and encouragement. Thanks to an associate editor and two referees for very helpful comments and suggestions. The paper has also benefitted from the suggestions of Masa Aoki, Susan Athey, Ales Filipi, Ali Hortacsu, Matthew Jackson, Tiefeng Jiang, Jonathan Levin, Howard Marvel, Leonardo Rezende, Ilya Segal, and Steve Tadelis. Financial support from Kapnick Dissertation Fellowship, Stanford Institute for Economic Policy Research, Stanford University is also gratefully acknowledged. All remaining errors are my own.

## Appendix A

**Proof of Proposition 3.** We consider the uniform-price auction first. By Eq. (4), if there exists any symmetric increasing equilibrium, the equilibrium bid function must be

$$\varphi^U(x) = E\Omega(x) - \hat{c}. \tag{11}$$

This shows the uniqueness. $\varphi^U(\cdot)$ is strictly increasing if and only if $E\Omega(\cdot)$ is strictly increasing. It remains to show that when $E\Omega(\cdot)$ is strictly increasing, (11) is indeed an equilibrium bid function.

Suppose all but bidder $i$ bid according to (11). If bidder $i$ bids $\varphi^U(x)$ while her true type is $x_i$, then her expected profit will be:

$$\begin{aligned}
\pi_i(x, x_i) &= E\mathbf{1}_{\varphi^U(x) > \varphi^U(X_{n;-i})}\big[\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) - \hat{c} - \varphi^U(X_{n;-i})\big] \\
&= E\mathbf{1}_{x > X_{n;-i}}\big[\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) - \hat{c} - \varphi^U(X_{n;-i})\big] \\
&= \int_0^x \big[E\big(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) \bigm| X_{n;-i} = t\big) - E\Omega(t)\big]\,\mathrm{d}F_{X_{n;-i}}(t).
\end{aligned}$$

Taking the derivative yields

$$\begin{aligned}
\frac{\partial \pi_i}{\partial x} &= \big[E\big(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) \bigm| X_{n;-i} = x\big) - E\Omega(x)\big] f_{X_{n;-i}}(x) \\
&= \big[E\big(\mathbf{1}_{x_i + Y_i > \widehat{S}_{1;-i}}\big(x_i + Y_i - \widehat{S}_{1;-i}\big) \bigm| X_{n;-i} = x\big) \\
&\quad - E\big(\mathbf{1}_{x + Y_i > \widehat{S}_{1;-i}}\big(x + Y_i - \widehat{S}_{1;-i}\big) \bigm| X_{n;-i} = x\big)\big] f_{X_{n;-i}}(x)
\end{aligned}$$

$$= \begin{cases} \geqslant 0 & x < x_i, \\ = 0 & x = x_i, \\ \leqslant 0 & x > x_i. \end{cases}$$

Therefore bidding according to (11) is also the optimal response for bidder $i$. This completes the proof of both the existence and the uniqueness under a uniform-price auction.

Now we consider the discriminatory-price auction.

First, by Lemma 2, in any symmetric increasing equilibrium, we have

$$\varphi^D(x_i) = \frac{1}{F_{X_{n;-i}}(x_i)} \int_0^{x_i} \varphi^U(x)\,\mathrm{d}F_{X_{n;-i}}(x),$$

where $\varphi^U(\cdot)$ is given by (11). Differentiating with respect to $x_i$, we have

$$\begin{aligned} \frac{\mathrm{d}\varphi^D(x_i)}{\mathrm{d}x_i} &= \frac{1}{F^2_{X_{n;-i}}(x_i)} \left[ \varphi^U(x_i) f_{X_{n;-i}}(x_i) F_{X_{n;-i}}(x_i) - f_{X_{n;-i}}(x_i) \int_0^{x_i} \varphi^U(x)\,\mathrm{d}F_{X_{n;-i}}(x) \right] \\ &= \frac{f_{X_{n;-i}}(x_i)}{F_{X_{n;-i}}(x_i)} \left[ \varphi^U(x_i) - E\big(\varphi^U(X_{n;-i}) \mid X_{n;-i} \leqslant x_i\big) \right]. \end{aligned}$$

$\varphi^U(x_i) - E(\varphi^U(X_{n;-i}) \mid X_{n;-i} \leqslant x_i) > 0$ if $\varphi^U(\cdot)$ is strictly increasing (since $X_{n;-i} = x_i$ with probability 0). Therefore, $\varphi^D(\cdot)$ is strictly increasing if $\varphi^U(\cdot)$ is strictly increasing or equivalently, if $E\Omega(\cdot)$ is strictly increasing.

Now following exactly the same steps as in the case of the uniform-price auction, we can show the uniqueness and existence under the discriminatory-price auction. □

**Proof of Proposition 4.** A bidder's expected gain from winning the second-stage auction conditional on being the marginal entrant in this case is

$$\begin{aligned} &E\big(\mathbf{1}_{S_i > \widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) \bigm| X_{n;-i} = x_i\big) \\ &\quad = E\big(\mathbf{1}_{x_i + Y_i > \widehat{S}_{1;-i}}\big(x_i + Y_i - \widehat{S}_{1;-i}\big) \bigm| X_{n;-i} = x_i\big) \\ &\quad = E\big(\mathbf{1}_{x_i + 1 > X_{1;-i}}(x_i + 1 - X_{1;-i}) \mid X_{n;-i} = x_i\big) \mathbf{1}_{Y_i = 1, Y_j = 0, j \in \mathbb{E} \setminus \{i\}} \\ &\quad = p(1-p)^{n-1} E\big((x_i + 1 - X_{1;-i}) \mid X_{n;-i} = x_i\big) \\ &\quad = V(1-p)^{n-2}\big[1 - E(X_{1;-i} - X_{n;-i} \mid X_{n;-i} = x_i)\big], \end{aligned}$$

where $V = p(1-p)$ is the variance of $Y$ ($\mathrm{Var}(Y)$), which is assumed to be positive in the current case.

We claim that $E(X_{1;-i} - X_{n;-i} \mid X_{n;-i} = x_i)$ is strictly decreasing in $x_i$ if $H(\cdot)$ is strictly increasing. Let $W_t$ be the random distance $X_{1;-i} - X_{n;-i}$ conditional on $X_{n;-i} = t$. To prove the claim, it suffices to show that $W_t$ is stochastically strictly decreasing in $t$ if $H'(\cdot) > 0$.

Treating densities as "probabilities," the density function of $W_t$ can be calculated as follows:

$$\begin{aligned} f_{W_t}(w) &= Pr(X_{1;-i} - X_{n;-i} = w \mid X_{n;-i} = t) \\ &= \frac{Pr(X_{1;-i} - X_{n;-i} = w, X_{n;-i} = t)}{Pr(X_{n;-i} = t)} \\ &= \frac{Pr(X_{1;-i} = w + t, X_{n;-i} = t)}{Pr(X_{n;-i} = t)} \end{aligned}$$

$$= \frac{\frac{(N-1)!}{(N-1-n)!(n-2)!}F^{N-1-n}(t)f(t)(F(w+t)-F(t))^{n-2}f(w+t)}{\frac{(N-1)!}{(N-1-n)!(n-1)!}F^{N-1-n}(t)f(t)(1-F(t))^{n-1}}$$
$$= (n-1)\frac{(F(w+t)-F(t))^{n-2}}{(1-F(t))^{n-1}}f(w+t).$$

The cumulative distribution function of $W_t$ can be computed as follows:

$$\begin{aligned} F_{W_t}(w) &= \int_0^w f_{W_t}(x)\,\mathrm{d}x \\ &= \int_0^w \frac{1}{(1-F(t))^{n-1}}\,\mathrm{d}\big[F(x+t)-F(t)\big]^{n-1} \\ &= \left[\frac{F(w+t)-F(t)}{1-F(t)}\right]^{n-1} \end{aligned}$$

where $w \in [0, 1-t]$.

Taking the derivative with respect to $t$, we have

$$\begin{aligned} \frac{\mathrm{d}F_{W_t}(w)}{\mathrm{d}t} &= (n-1)\left[\frac{F(w+t)-F(t)}{1-F(t)}\right]^{n-2}\frac{1}{(1-F(t))^2}\big\{\big[f(w+t)-f(t)\big]\big(1-F(t)\big) \\ &\quad + \big(F(w+t)-F(t)\big)f(t)\big\} \\ &= (n-1)\left[\frac{F(w+t)-F(t)}{1-F(t)}\right]^{n-2}\frac{1-F(w+t)}{1-F(t)}\big[H(w+t)-H(t)\big]. \end{aligned}$$

Therefore, $\frac{\mathrm{d}F_{W_t}(w)}{\mathrm{d}t} > 0$ for any $w \in [0, 1-t]$ if $H'(\cdot) > 0$. This proves the claim. The result follows from the claim and Proposition 3. □

**Proof of Proposition 5.** By Lemma 2, in any symmetric increasing equilibrium, we have

$$\varphi^A(x) = \int_0^x \varphi^U(t)\,\mathrm{d}F_{X_{n;-i}}(t), \tag{12}$$

where $\varphi^U(\cdot)$ is given in (11). So $\varphi^A(x)$ is strictly increasing if and only if $\varphi^U(x)$ is strictly positive. But this can always be achieved by choosing a sufficiently large $K$, if necessary. Using a similar argument in the proof of Proposition 3, we can show that such a function is indeed a unique equilibrium bid function. □

**Proof of Proposition 7.** Let us only derive the expected revenue for the private value updating case with $n \geqslant 2$. The case of common value updating can be shown analogously.

By IC condition (4) and the boundary condition $m_i^{\mathcal{F}}(0) = 0$, we have

$$m_i^{\mathcal{F}}(x_i) = \int_0^{x_i} E\Omega(x)\,\mathrm{d}F_{X_{n;-i}}(x) - \hat{c}F_{X_{n;-i}}(x_i).$$

Substituting the expression of $m_i^{\mathcal{F}}(x_i)$ into (3) evaluated at $x_i' = x_i$, we have

$$\pi_i(x_i) = E\mathbf{1}_{X_{n;-i}<x_i}\mathbf{1}_{S_i>\widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) - \hat{c}F_{X_{n;-i}}(x_i) - m^{\mathcal{F}}(x_i)$$
$$= E\mathbf{1}_{X_{n;-i}<x_i}\mathbf{1}_{S_i>\widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big) - \int_0^{x_i} E\Omega(x)\,\mathrm{d}F_{X_{n;-i}}(x).$$

The bidders' expected total profit is:

$$E\Pi = NE\pi_i(X_i)$$
$$= NE_{X_i}\big[E\mathbf{1}_{X_{n;-i}<X_i}\mathbf{1}_{S_i>\widehat{S}_{1;-i}}\big(S_i - \widehat{S}_{1;-i}\big)\big] - EM$$
$$= E\big(\widehat{S}_{1;n} - \widehat{S}_{2;n}\big) - EM.$$

Therefore, the expected revenue is:

$$ER = E\widehat{S}_{1;n} - nc - E\Pi = E\widehat{S}_{2;n} + EM - nc. \qquad \square$$

Table 1
Optimal numbers of second-stage bidders (private value updating case)

| | $c/\overline{V}$ <br> $n$ | $E\widehat{S}_{2;n}$ | $EM$ | 0.0001 <br> $ER$ | 0.0005 <br> $ER$ | 0.001 <br> $ER$ | 0.002 <br> $ER$ | 0.005 <br> $ER$ | 0.01 <br> $ER$ |
|---|---|---|---|---|---|---|---|---|---|
| $N=100$ | 1 | | | 1.4800 | 1.4792 | 1.4782 | 1.4762 | 1.4702 | 1.4602 |
| $p=0.5$ | 2 | 1.2382 | 0.4926 | 1.7304 | 1.7288 | 1.7268 | 1.7228 | 1.7108 | 1.6908 |
| | 3 | 1.4759 | 0.3651 | 1.8404 | 1.8380 | 1.8350 | 1.8290 | 1.8110 | 1.7810 |
| | 4 | 1.6581 | 0.2407 | 1.8980 | 1.8948 | 1.8908 | 1.8828 | 1.8588 | 1.8188 |
| | 5 | 1.7849 | 0.1488 | 1.9328 | 1.9288 | 1.9238 | 1.9138 | 1.8838 | **1.8338** |
| | 6 | 1.8619 | 0.0883 | 1.9490 | 1.9442 | 1.9382 | 1.9262 | **1.8902** | 1.8302 |
| | 7 | 1.8996 | 0.0510 | 1.9492 | 1.9436 | 1.9366 | 1.9226 | 1.8806 | 1.8106 |
| | 8 | 1.9305 | 0.0288 | 1.9577 | **1.9515** | **1.9435** | **1.9275** | 1.8795 | 1.7995 |
| | 9 | 1.9420 | 0.0160 | 1.9563 | 1.9491 | 1.9401 | 1.9221 | 1.8681 | |
| | 10 | 1.9504 | 0.0088 | 1.9572 | 1.9492 | 1.9392 | | | |
| | 11 | 1.9553 | 0.0048 | **1.9579** | 1.9491 | | | | |
| | 12 | 1.9575 | 0.0026 | 1.9577 | | | | | |
| $N=100$ | 1 | | | 1.7800 | 1.7792 | 1.7782 | 1.7762 | 1.7702 | 1.7602 |
| $p=0.8$ | 2 | 1.6163 | 0.3152 | 1.9311 | 1.9295 | 1.9275 | 1.9235 | 1.9115 | 1.8915 |
| | 3 | 1.8736 | 0.0935 | 1.9664 | 1.9640 | 1.9610 | 1.9550 | **1.9370** | **1.9070** |
| | 4 | 1.9492 | 0.0246 | 1.9730 | **1.9698** | **1.9658** | **1.9578** | 1.9338 | 1.8938 |
| | 5 | 1.9683 | 0.0061 | 1.9733 | 1.9693 | 1.9643 | 1.9543 | 1.9243 | 1.8743 |
| | 6 | 1.9741 | 0.0014 | **1.9743** | 1.9695 | 1.9635 | 1.9515 | 1.9155 | 1.8555 |
| | 7 | 1.9748 | 0.0003 | 1.9737 | 1.9681 | 1.9611 | 1.9471 | 1.9051 | 1.8351 |
| $N=50$ | 1 | | | 1.4606 | 1.4598 | 1.4588 | 1.4568 | 1.4508 | 1.4408 |
| $p=0.5$ | 2 | 1.2120 | 0.4853 | 1.6969 | 1.6953 | 1.6933 | 1.6893 | 1.6773 | 1.6573 |
| | 3 | 1.4642 | 0.3554 | 1.8190 | 1.8166 | 1.8136 | 1.8076 | 1.7896 | 1.7596 |
| | 4 | 1.6422 | 0.2316 | 1.8730 | 1.8698 | 1.8658 | 1.8578 | 1.8338 | **1.7938** |
| | 5 | 1.7517 | 0.1415 | 1.8923 | 1.8883 | 1.8833 | 1.8733 | 1.8433 | 1.7933 |
| | 6 | 1.8282 | 0.0830 | 1.9101 | 1.9053 | 1.8993 | 1.8873 | **1.8513** | 1.7913 |
| | 7 | 1.8711 | 0.0473 | 1.9171 | 1.9115 | **1.9045** | **1.8905** | 1.8485 | 1.7785 |
| | 8 | 1.8930 | 0.0264 | 1.9178 | 1.9114 | 1.9034 | 1.8874 | 1.8394 | |
| | 9 | 1.9079 | 0.0145 | 1.9206 | **1.9134** | 1.9044 | 1.8864 | 1.8324 | |
| | 10 | 1.9151 | 0.0079 | **1.9209** | 1.9129 | 1.9029 | 1.8829 | | |
| | 11 | 1.9173 | 0.0042 | 1.9194 | 1.9106 | | | | |

(*continued on next page*)

Table 1 (*continued*)

| | $c/\overline{V}$ <br> $n$ | $E\widehat{S}_{2;n}$ | $EM$ | 0.0001 <br> $ER$ | 0.0005 <br> $ER$ | 0.001 <br> $ER$ | 0.002 <br> $ER$ | 0.005 <br> $ER$ | 0.01 <br> $ER$ |
|---|---|---|---|---|---|---|---|---|---|
| $N=50$ | 1 | | | 1.7606 | 1.7598 | 1.7588 | 1.7568 | 1.7508 | 1.7408 |
| $p=0.8$ | 2 | 1.6017 | 0.3106 | 1.9119 | 1.9103 | 1.9083 | 1.9043 | 1.8923 | 1.8723 |
| | 3 | 1.8592 | 0.0910 | 1.9496 | **1.9472** | **1.9442** | **1.9382** | **1.9202** | **1.8902** |
| | 4 | 1.9275 | 0.0237 | **1.9504** | **1.9472** | 1.9432 | 1.9352 | 1.9112 | 1.8712 |
| | 5 | 1.9441 | 0.0058 | 1.9489 | 1.9449 | 1.9399 | 1.9299 | 1.8999 | 1.8499 |

The maximal total value $\overline{V}=2$. The expected revenue is computed according to the following formula: $ER = EX_{2;n} + EY - c$ if $n=1$, and $ER = E\widehat{S}_{2;n} + EM - nc$ if $n \geqslant 2$. Given each entry cost level $c/\overline{V}$, the optimal number of bidders is the $n$ that corresponds to the maximal expected revenue in boldface.

Table 2
Optimal numbers of second-stage bidders (common value updating case)

| | $c/\overline{V}$ <br> $n$ | $E\widehat{S}_{2;n}$ | $EM$ | 0.0001 <br> $ER$ | 0.0005 <br> $ER$ | 0.001 <br> $ER$ | 0.002 <br> $ER$ | 0.005 <br> $ER$ | 0.01 <br> $ER$ |
|---|---|---|---|---|---|---|---|---|---|
| $N=100$ | 1 | | | 1.4800 | 1.4792 | 1.4782 | 1.4762 | 1.4702 | 1.4602 |
| $p=0.5$ | 2 | 0.9841 | 0.0927 | **1.5714** | **1.5698** | **1.5678** | **1.5638** | **1.5518** | **1.5318** |
| | 3 | 0.9851 | 0.0154 | 1.4948 | 1.4924 | 1.4894 | 1.4834 | 1.4654 | 1.4354 |
| | 4 | 0.9854 | 0.0019 | 1.4815 | 1.4783 | 1.4743 | 1.4663 | 1.4423 | 1.4023 |
| $N=100$ | 1 | | | 1.7800 | 1.7792 | 1.7782 | 1.7762 | 1.7702 | 1.7602 |
| $p=0.8$ | 2 | 0.9876 | 0.0593 | **1.8385** | **1.8369** | **1.8349** | **1.8309** | **1.8189** | **1.7989** |
| | 3 | 0.9883 | 0.0039 | 1.7836 | 1.7812 | 1.7782 | 1.7722 | 1.7542 | 1.7242 |
| | 4 | 0.9882 | 0.0002 | 1.7796 | 1.7764 | 1.7724 | 1.7644 | 1.7404 | 1.7004 |
| $N=50$ | 1 | | | 1.4606 | 1.4598 | 1.4588 | 1.4568 | 1.4508 | 1.4408 |
| $p=0.5$ | 2 | 0.9687 | 0.0934 | **1.5517** | **1.5501** | **1.5481** | **1.5441** | **1.5321** | **1.5121** |
| | 3 | 0.9709 | 0.0154 | 1.4606 | 1.4732 | 1.4702 | 1.4642 | 1.4462 | 1.4162 |
| | 4 | 0.9709 | 0.0020 | 1.4620 | 1.4588 | 1.4548 | 1.4468 | 1.4228 | 1.3828 |
| $N=50$ | 1 | | | 1.7606 | 1.7598 | 1.7588 | 1.7568 | 1.7508 | 1.7408 |
| $p=0.8$ | 2 | 0.9757 | 0.0598 | **1.8191** | **1.8175** | **1.8155** | **1.8115** | **1.7995** | **1.7795** |
| | 3 | 0.9769 | 0.0039 | 1.7642 | 1.7618 | 1.7588 | 1.7528 | 1.7348 | 1.7048 |
| | 4 | 0.9774 | 0.0002 | 1.7608 | 1.7576 | 1.7536 | 1.7456 | 1.7216 | 1.6816 |

The maximal total value $\overline{V}=2$. The expected revenue is computed according to the following formula: $ER = EX_{2;n} + EY - c$ if $n=1$, and $ER = E\widehat{S}_{2;n} + \frac{N-1}{N}EY + EM - nc$ if $n \geqslant 2$. Given each entry cost level $c/\overline{V}$, the optimal number of bidders is the $n$ that corresponds to the maximal expected revenue in boldface.

## References

Barzel, Y., 1982. Measurement cost and the organization of markets. J. Law Econ. 25, 27–48.
Bikhchandani, S., Riley, J., 1991. Equilibria in open common value auctions. J. Econ. Theory 53, 101–130.
Bulow, J., Huang, M., Klemperer, P., 1999. Toeholds and takeovers. J. Polit. Economy 107, 427–454.
Bulow, J., Klemperer, P., 2002. Prices and the winner's curse. RAND J. Econ. 33, 1–21.
Engelbrecht-Wiggans, R., 1993. Optimal auctions revisited. Games Econ. Behav. 5, 227–239.
French, R.K., McCormick, R.E., 1984. Sealed bids, sunk costs, and the process of competition. J. Bus. 57, 417–441.
Fullerton, R., McAfee, P., 1999. Auctioning entry into tournaments. J. Polit. Economy 107, 573–605.
Goeree, J., Offerman, T., 2003. Competitive bidding in auctions with private and common values. Econ. J. 113, 598–613.
Harris, M., Raviv, A., 1981. Allocation mechanisms and the design of auctions. Econometrica 49, 1477–1499.
Jackson, M., Simon, L., Swinkels, J., Zame, W., 2002. Communication and equilibrium in discontinuous games of incomplete information. Econometrica 70, 1711–1740.

Johnson, N.R., 1979. Auction markets, bid preparation costs and entrance fees. Land Econ. 55, 313–317.
Kagel, J., 1995. Auctions: A survey of experimental research. In: Kagel, J., Roth, A. (Eds.), The Handbook of Experimental Economics. Princeton Univ. Press, pp. 501–586.
Klemperer, P., 1998. Auctions with almost common values: The 'wallet game' and its applications. Europ. Econ. Rev. 42, 757–769.
Krishna, V., Morgan, J., 1997. (Anti-) Competitive effects of joint bidding and bidder restrictions. Working paper. Penn State University and Princeton University.
Leffler, K., Rucker, R., Munn, I., 2000. Transaction costs and the collection of information: Presale measurement on private timber sales. J. Law, Econ. Organiz. 16, 166–188.
Levin, D., Smith, J., 1994. Equilibrium in auctions with entry. Amer. Econ. Rev. 84, 585–599.
Maskin, E., Riley, J., 1989. Optimal multi-unit auctions. In: Hahn, F. (Ed.), The Economics of Missing Markets, Information, and Games. Clarendon, Oxford, pp. 312–335.
McAdams, D., Fujishima, Y., Shoham, Y., 1999. Speeding up ascending-bid auctions. Proceedings of the International Journal Conference of Artificial Intelligence.
McAfee, P., McMillan, J., 1987. Auctions with entry. Econ. Letters 23, 343–347.
Milgrom, P., 2004. Putting Auction Theory to Work (Churchill Lectures in Economics). Cambridge Univ. Press.
Milgrom, P., Weber, R., 1982. A theory of auctions and competitive bidding. Econometrica 50, 1082–1122.
Myerson, R., 1981. Optimal auction design. Math. Operations Res. 6, 58–73.
Perry, M., Wolfstetter, E., Zamir, S., 2000. A sealed-bid auction that matches the English auction. Games Econ. Behav. 33, 265–273.
Riley, J., Samuelson, W., 1981. Optimal auctions. Amer. Econ. Rev. 71, 381–392.
Rothkopf, M., 1969. A model of rational competitive bidding. Manage. Sci. 15, 362–373.
Samuelson, W., 1985. Competitive bidding with entry costs. Econ Letters 17, 53–57.
Simon, L., Zame, W., 1990. Discontinuous games and endogenous sharing rules. Econometrica 58, 861–872.
Vallen, M., Bullinger, C., 1999. The due diligence process for acquiring and building power plants. Electricity J. (October), 28–37.
Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders. J. Finance 16, 8–37.
Weber, R., 1983. Multi-object auctions. In: Engelbrecht-Wiggans, R., Shubik, M., Stark, R.M. (Eds.), Auction, Bidding, and Contracting: Uses and Theory. New York Univ. Press, New York, pp. 165–194.
Wilson, R., 1977. A bidding model of perfect competition. Rev. Econ. Stud. 44, 511–518.
Ye, L., 2001. Auctions with costly entry. PhD dissertation. Stanford University.
Ye, L., 2004. Optimal auctions with endogenous entry. Contrib. Theoretical Econ. 4, 1–27.