Auctions with Entry and Resale: The Case with Continuous Heterogeneous Entry Costs

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Abstract

We study how resale affects auctions with costly entry in a model where bidders possess two-dimensional private information signals: entry costs and valuations. We establish the existence of symmetric entry equilibrium and identify sufficient conditions under which the equilibrium is unique. Our analysis suggests that the opportunity of resale induces motivation for both speculative entry and bargain hunting abstentions. By following a distribution family allowing for any degree of correlation between entry costs and valuations, we compare the equilibrium with resale to the equilibrium without resale. Our results suggest that while efficiency is always higher when resale is allowed, the implications for entry probability and expected revenue are ambiguous, which, in particular, depends on the reseller's bargaining power in the resale stage.

1 Introduction

Starting with the seminal work of Vickrey (1961), the auction literature has mostly adopted the paradigm of a fixed number of bidders. This simplifying assumption allows for enormous progress in the characterization of optimal auctions and analysis on revenue equivalence and revenue rankings of different auction formats (see, for example, Riley and Samuelson (1981), Myerson (1981), Milgrom

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and Weber (1982)). However, in many auctions the number of rivals is not known when bids are placed.[1] This observation motivated several papers (e.g. McAfee and McMillan (1987), Mathews (1987), and Harstad, Kagel, and Levin (1990)) to treat the number of rival bidders as coming from exogenously determined distributions. In models with such an extension, bidders may have preferences over different auction formats, which in turn has important impact on optimal auction designs. In addition, auctioneers may have one more design instrument regarding whether to reveal or to conceal the number of bidders.[2] However, this approach with a stochastic number of bidders still ignores an important issue. In many auctions the cost of bid preparation or information acquisition is far from trivial. We cannot simply rank auctions by assuming a fixed or stochastic number of bidders without accounting for the fact that different auctions are likely to induce different entry incentives. Endogenous entry must be taken into account in order to compare expected revenue or to design optimal auctions in those situations.

These considerations have motivated a growing literature on auctions with costly entry.[3] Independent-private-value (IPV) auctions with costly entry often result in inefficient allocation, since the auction is typically conducted among the actual bidders, which is a subset of all potential bidders. If the bidder with the highest value is excluded from the auction, the auction outcome is necessarily inefficient *ex post*.

This sort of inefficiency may create a motive for post-auction resale and such resale opportunity affects significantly, as we shall show, bidders' bidding behavior and entry strategies. In this paper, we study the effect of allowing resale opportunity in an auction model where each potential bidder possesses two-dimensional private information signals: entry cost (c) and valuation (v). When the opportunity of resale is absent, this framework is first analyzed by Green and Laffont (1984), who demonstrate that under a Vickrey auction the entry equilibrium is characterized by a unique entry cutoff curve (or entry indifference curve) $C(\cdot)$ such that a bidder with type (c, v) enters the auction

[1] This is the case, for example, in most of the sealed-bid procurement auctions.

[2] See Harstad et al. for how to implement such a scheme even when the seller does not know *ex ante* the actual number of bidders. Also see Levin and Ozdenren (2004) who revisit these questions under ambiguity.

[3] See, among others, French and McCormick (1984), Green and Laffont (1984), Samuelson (1985), McAfee and McMillan (1987), Tan (1992), Engelbrecht-Wiggans (1993), Levin and Smith (1994), Stegeman (1996), Tan and Yilankaya (2006), Gal et al. (2007), Ye (2004, 2007), Lu (2006, 2007), and Moreno and Wooders (2008). Also see Bergemann and Välimäki (2006) for an extensive survey of the literature.

if and only if $c \leq C(v)$. In such an equilibrium, it is possible that a bidder with a high value chooses not to enter the auction, simply because her entry cost is above her entry cutoff. Following Green and Laffont's analysis, Lu (2006) characterizes the symmetric entry equilibrium cutoff curves implemented by the classes of *ex post* efficient or *ex post* revenue-maximizing mechanisms. In a general procurement setting allowing for correlations between participation costs and production costs, Gal, Landsberger, and Nemirovski (2007) establish the existence and uniqueness of the equilibrium entry cutoff curve.[4]

Introducing post-auction resale into models with costly entry enriches, yet complicates the model so that even the existence and uniqueness of entry equilibrium are no longer obvious. The additional difficulty emerges since with resale, the option value for staying out is also positive, and varies with types. We nevertheless show in this paper that a symmetric entry equilibrium is still characterized by an entry cutoff curve, and we identify sufficient conditions that assure the uniqueness of equilibrium. Integrating resale into two-dimensional model of auction with entry is quite challenging and our analysis is done under two important simplifications. First, we focus on the case with two bidders, and second, we assume that the resale is conducted under complete information. The restrictions of these simplifications are discussed in Section 5.

We compare the equilibrium with resale to the equilibrium without resale. Our first finding is that with resale, the entry cutoff is higher for types with sufficiently low values, and lower for types with sufficiently high values. This suggests that when resale is allowed, bidders with low values are more likely to enter, while bidders with high values are less likely to enter. In other words, resale naturally induces a *speculative motivation* for entry and *bargain hunting* motivation for staying out. Speculators are those with low entry costs and low valuations who would not enter without resale, but enter when resale opportunity is available. Bargain hunters are those with high entry costs and high valuations who would refrain from entering when a resale market is available, but enter when resale is unavailable. Thus, although we may be tempted to think that the additional opportunity to trade in the resale market provides an additional motive to enter, the opposite incentive to avoid entry cost by staying out and buying in the resale market may more than offset it.

Working with a specific, yet flexible, distribution family which allows for any degree of correlation

[4] Moreover, they demonstrate that providing participation reimbursements can partially mitigate the problem with entry and improve the seller's expected revenue.

between c and v, we obtain more precise comparison results. In the benchmark case where c and v are independently and uniformly distributed, we show that there is a unique turning point v^* such that when $v < v^*$, the entry cutoff curve with resale lies above the entry cutoff curve without resale; while when $v > v^*$, the reverse holds. So the entry comparison result is made stronger in this case: resale induces more entry for a bidder if her value is lower than some threshold, and induces less entry if her value is higher than that threshold.

Under different parameter values about the correlation between c and v and relative bargaining power between the reseller (i.e., the auction winner) and buyer, we compare the equilibrium entry probability, expected revenue, and expected surplus in the resale and no resale cases. In all the cases we consider, resale leads to higher expected efficiency, which is consistent with our general notion that allowing resale should help correct the inefficiency induced by costly entry. However, the effects of resale on expected entry and expected revenue (for the original item owner) are ambiguous depending on the relative bargaining power between the reseller and buyer in the resale stage.

The literature on auctions with resale is relatively new. Gupta and Lebrun (1999) consider first-price auctions with resale, where complete information is assumed in the resale stage. Haile (2003) considers an IPV setup in which bidders only have noisy signals at the auction stage, where the motive for resale arises when the true value of the auction winner turns out to be low. Zheng (2002) identifies conditions under which the outcome of optimal auctions can be achieved with resale. Garret and Tröger (2006) consider a model with a speculator, who can only benefit from participating in the auction when she can resell the item to the other bidder. Pagnozzi (2007) demonstrates why a strong bidder may prefer to drop out of an auction before the price reaches her valuation, simply because she anticipates that she will be in an advantageous position in the post-auction resale stage. Hafalir and Krishna (2008) analyze auctions with resale in an asymmetric IPV auction environment with two bidders and show that the expected revenue is higher under a first-price auction than it is in a second-price auction. Finally, Garratt, Tröger, and Zheng (2009) show that when resale is allowed, the English ascending auction is susceptible to tacit collusion. They construct equilibria that interim Pareto dominate the standard truthful value-bidding equilibrium. Our paper contributes to the literature by being the first to integrate and analyze both entry and resale in an auction model with two-dimensional private information.

The paper is organized as follows. Section 2 lays out the model. Section 3 characterizes the equilibrium and derives sufficient conditions under which the equilibrium is unique. Section 4 is devoted to comparisons of the equilibria in the resale and no resale cases. Section 5 concludes.

2 The Model

There is a single, indivisible object for sale to 2 potentially interested buyers (firms) through an auction. The seller's valuation is normalized to 0. It is costly for a bidder to participate in the auction (e.g., it is costly to prepare blueprints, to hire experts for best bidding strategies, or simply to establish eligibility to bid or to meet some legal requirements, etc.). So unlike most auction models, in our model each bidder possesses two-dimensional private information about her participation cost (c) and value (v). We assume that c and v follow a joint distribution $H(c, v)$ on $[0, 1] \times [0, 1]$, with a continuously differentiable density function $h(c, v)$. We assume that a second-price sealed-bid auction is conducted, and bidders' evaluations are private. For simplicity of analysis, we assume that the seller does not set a reserve price other than his own reservation value 0. After the auction, the winner can conduct a post-auction resale.[5] For ease of analysis, we assume that resale occurs under complete information and the outcome of bargaining between the reseller (i.e., the auction winner) and buyer is given by the Nash bargaining solution.[6] That is, whenever there is a potential gain from resale, the reseller gets λ portion and the buyer gets $1 - \lambda$ portion of the surplus, where $\lambda \in [0, 1]$. The parameter λ thus captures the reseller's relative bargaining power in the resale stage. When $\lambda = 1/2$, i.e., when the reseller and buyer have identical bargaining power, the outcome corresponds to the symmetric Nash bargaining solution.

We assume that the seller makes the selling mechanism publicly known before participation (or entry) occurs. After learning the realizations of their private entry costs and values, the potential bidders make entry decisions simultaneously and independently. After entry, the bidders bid for the item. We will consider the two scenarios when resale is allowed and not allowed.

[5] There is no obvious reason to believe that the heterogenous entry costs that we envision also apply to the resale. We thus assume that the entry costs for the resale market are the same. We further assume that this cost is small, and is hence normalized to be zero.

[6] Gupta and Lebrun (1999) and Pagnozzi (2007) also assume complete information in the resale stage.

3 Equilibrium Characterizations

In this section we characterize equilibria in the no-resale benchmark case and in the case when resale is allowed. We will focus on the symmetric entry equilibrium characterized by a twice continuously differentiable entry cutoff curve or entry indifference curve, $\tilde{C}(\cdot)$ in the no resale case and $C(\cdot)$ in the case with resale, so that a bidder with a type (c, v) enters the auction if and only if $c \leq \tilde{C}(v)$ in the no resale benchmark case and $c \leq C(v)$ in the case with resale.

3.1 Entry without Resale

Since bidders are *ex ante* symmetric, without loss of generality we can focus on bidder 1's entry strategy. To show that $\tilde{C}(\cdot) : [0,1] \longrightarrow [0,1]$ characterizes a symmetric entry equilibrium, we need to verify that there is no incentive for bidder 1 to deviate from the prescribed entry strategy given that bidder 2 follows the same strategy.

When bidder 2 follows the entry cutoff curve $\tilde{C}(\cdot)$, the probability that she enters the auction is given by:

$$\tilde{q} = \int_0^1 \int_0^{\tilde{C}(\xi)} dH(\eta, \xi).$$

From bidder 1's perspective, $\tilde{F}_{in}(x) = \left[\int_0^x \int_0^{\tilde{C}(\xi)} dH(\eta, \xi)\right] / \tilde{q}$ is the probability that bidder 2 has a value less than x conditional on entry, and $\tilde{f}_{in}(x) = \left[\int_0^{\tilde{C}(x)} h(\eta, x) d\eta\right] / \tilde{q}$ is the associated density of a value equal to x.

Similarly, $\tilde{F}_{out}(x) = \left[\int_0^x \int_{\tilde{C}(\xi)}^1 dH(\eta, \xi)\right] / (1 - \tilde{q})$ is the probability that bidder 2 has a value less than x, conditional on staying out, and $\tilde{f}_{out}(x) = \left[\int_{\tilde{C}(x)}^1 h(\eta, x) d\eta\right] / (1 - \tilde{q})$ is the associated density of a value x.

We assume that each bidder plays the (weakly) dominant strategy to bid her value after entry.[7] If bidder 1 with a value v enters, her payoff (from the auction alone) depends on whether bidder 2 enters. If bidder 2 stays out (with probability $1-\tilde{q}$), her payoff is v (since we assume that the reserve price is 0); if bidder 2 enters (with probability $\tilde{q}$), her expected payoff is given by $\int_0^v (v - \xi) d\tilde{F}_{in}(\xi) = \int_0^v \tilde{F}_{in}(\xi) d\xi$. Hence bidder 1's expected payoff from entering the auction, given that bidder 2 follows entry equilibrium $\tilde{C}(\cdot)$, is given by:

[7] Note that participation costs are sunk costs, which should not affect the bidding strategies.

$$E\Pi(v) = \tilde{q}\int_0^v \tilde{F}_{in}(\xi)d\xi + (1-\tilde{q})v.$$

It can be easily verified that $E\Pi(v)$ is continuous, differentiable, and strictly increasing in v. In particular, $E\Pi(0) = 0$ and $E\Pi(1) \leq 1$.

For $\tilde{C}(\cdot)$ to constitute a symmetric entry cutoff curve, we must have the following indifference condition:

$$\tilde{C}(v) = \tilde{q}\int_0^v \tilde{F}_{in}(\xi)d\xi + (1-\tilde{q})v \tag{1}$$

That is, if bidder 1's cost lies exactly on the entry cutoff curve (i.e., $c = \tilde{C}(v)$), she should be indifferent between entering the auction and staying out.

Note that $E\Pi(v) \geq c$ if and only if $c \leq \tilde{C}(v)$. Thus, given that bidder 2 follows $\tilde{C}(\cdot)$, it is also bidder 1's best response to follow $\tilde{C}(\cdot)$. This implies that the solution of $\tilde{C}(\cdot)$ to (1) indeed constitutes a symmetric entry equilibrium.

Differentiating (1), we have:

$$\tilde{C}'(v) = \tilde{q}\tilde{F}_{in}(v) + (1-\tilde{q}) \tag{2}$$

It can be easily verified that $\tilde{C}(0) = 0$, and from (2) that $\tilde{C}'(1) = 1$. Adapting the proof in Gal et al. (2007), we can establish the following proposition.[8]

Proposition 1 *There is a unique solution, $\tilde{C}(\cdot)$, to the following differential equation system, which characterizes the entry equilibrium when resale is absent:*

$$\begin{cases} \tilde{C}''(v) = \tilde{q}\tilde{f}_{in}(v) = \int_0^{\tilde{C}(v)} h(\eta, v)d\eta \\ \tilde{C}(0) = 0 \\ \tilde{C}'(1) = 1 \end{cases}$$

The general case of the above differential equation system cannot be solved explicitly. One exception is when both c and v follow a uniform distribution (the solution is given in Section 4).

[8] The assumption that $h(\eta, v)$ is continuously differentiable is needed in the proof.

3.2 Entry with Resale

We now augment the model analyzed in the previous section by allowing a resale stage in which the auction winner may resell the item. Similarly to the benchmark case without resale, to demonstrate that $C(\cdot) : [0,1] \longrightarrow [0,1]$ characterizes a symmetric entry equilibrium, we need to verify that there is no incentive for bidder 1 to deviate from this prescribed entry strategy given that bidder 2 follows the same strategy.

When bidder 2 follows the entry cutoff curve $C(\cdot)$, the (*ex ante*) probability that she enters the auction is given by:

$$q = \int_0^1 \int_0^{C(\xi)} dH(\eta, \xi). \tag{3}$$

In the following analysis, we assume that the bargaining power parameter in the resale stage, λ, is fixed; the reseller and buyer split the surplus (if any) so that the reseller obtains λ portion of the surplus and the buyer obtains $(1-\lambda)$ portion of the surplus.

Since we only consider two potential bidders, bidding one's own value remains to be an equilibrium.[9] Suppose $F_{in}(\cdot), f_{in}(\cdot), F_{out}(\cdot)$,and $f_{out}(\cdot)$ are all defined analogously as in the previous section. Suppose the value bidder 1 possesses is v. When she enters the auction, the expected payoff (without considering entry cost) is now given by:

$$q \int_0^v F_{in}(t)dt + (1-q)v + \lambda \int_v^1 (\xi - v) f_{out}(\xi) d\xi.$$

The first two terms are the same as in the case without resale. But the additional term reflects the potential gain from resale, which occurs when bidder 2, with a value higher than v, does not enter.

If bidder 1 decides not to enter the auction, she still has a chance to obtain the item through resale. This happens when bidder 2 enters with a value lower than v. In this case, the expected gain from resale is given by

[9] This is no longer the case when we consider more than two bidders. When the number of bidders $n > 2$, the probability of resale is strictly greater than zero even when two or more bidders enter the auction. But then, bidding one's own value no longer constitutes an equilibrium, as bidders should take into account the continuation values induced by the likely post-auction sale. The major cost of restricting our analysis to a two-bidder case is the lack of this effect of resale on bidding strategies. We will come back to this in Section 5.

$$q\int_0^v (1-\lambda)(v-\xi)f_{in}(\xi)d\xi.$$

Let $E\Pi(v)$ denote the expected net gain from entry over not entering (net of entry cost). Then we have:

$$\begin{aligned} E\Pi(v) &= q\int_0^v F_{in}(t)dt + (1-q)\left[\int_0^v v f_{out}(\xi)d\xi + \int_v^1 [\lambda\xi + (1-\lambda)v] f_{out}(\xi)d\xi\right] \\ &\quad -q\int_0^v (1-\lambda)(v-\xi)f_{in}(\xi)d\xi. \end{aligned}$$

It can be verified that $E\Pi(v)$ is continuous, differentiable, and strictly increasing in v. So the higher the value, the higher the expected gain from entry compared to non-entry.

Following the arguments paralleling those in the previous section, we can conclude that any solution $C(\cdot)$ to the following indifference condition, should it exist, constitutes a symmetric entry equilibrium:

$$\begin{aligned} C(v) &= q\int_0^v F_{in}(t)dt + (1-q)\left[\int_0^v v f_{out}(\xi)d\xi + \int_v^1 [\lambda\xi + (1-\lambda)v] f_{out}(\xi)d\xi\right] \\ &\quad -q\int_0^v (1-\lambda)(v-\xi)f_{in}(\xi)d\xi. \end{aligned} \tag{4}$$

Differentiating (4), we have:

$$C'(v) = \lambda H(1,v) + (1-\lambda)(1-q). \tag{5}$$

Let c_0 denote the initial value $C(0)$. Integrating (5) from 0 to v, we have:

$$C(v) = c_0 + (1-\lambda)(1-q)v + \lambda\int_0^v H(1,t)dt. \tag{6}$$

Substituting $v = 0$ into (4)), we have:

$$c_0 = \lambda\int_0^1\int_{C(\xi)}^1 \xi h(\eta,\xi)d\eta d\xi. \tag{7}$$

The initial value of the entry cutoff curve, c_0, is the equilibrium expected payoff conditional on entry for a bidder with $v = 0$; such a bidder will earn zero payoff conditional on staying out, and when she enters, she can only win if the other bidder stays out, in which case she obtains λ portion of the resale surplus.

The existence of a symmetric entry equilibrium boils down to the existence of a solution to the system of equations (3), (6), and (7). In what follows we will show that such a solution exists and under certain conditions the solution (and hence the equilibrium) is also unique.

Plugging (6) into (3) we can obtain an equation involving c_0 and q. Define

$$\gamma(q, c_0) = \int_0^1 \int_0^{C(\xi)} dH(\eta, \xi) - q,$$

where $C(\xi)$ is given by (6). Clearly, $\gamma(q, c_0)$ is continuously differentiable, and $\gamma(0, c_0) > 0$, $\gamma(1, c_0) < 0$. Therefore, for any c_0, there is at least one q_0 such that $\gamma(q_0, c_0) = 0$. We will show that q_0 is uniquely determined and there is a continuously differentiable function $g(\cdot)$ such that $q_0 = g(c_0)$. Based on this, we will argue that a symmetric equilibrium must exist. Moreover, the search for sufficient conditions under which the symmetric entry equilibrium is unique can be facilitated by defining the following function:

$$\begin{aligned}\Omega(c_0, q) \quad = \quad & 1 + \int_0^1 \xi h\left(\phi(c_0, q, \xi), \xi\right) d\xi + \lambda(1-\lambda)\left[\int_0^1 \xi h\left(\phi(c_0, q, \xi), \xi\right) d\xi\right]^2 \\ & -\lambda(1-\lambda)\int_0^1 \xi^2 h\left(\phi(c_0, q, \xi), \xi\right) d\xi \int_0^1 h\left(\phi(c_0, q, \xi), \xi\right) d\xi, \end{aligned} \tag{8}$$

where $\phi(c_0, q, \xi) = c_0 + (1-\lambda)(1-q)v + \lambda \int_0^v H(1, t)dt$.[10]

Proposition 2 *When resale is allowed, the system of equations (3), (6), and (7) admits at least one solution of* $C(\cdot)$*; any solution constitutes a symmetric entry equilibrium with resale. Moreover, the solution (and hence the symmetric entry equilibrium) is unique if for all* $(c_0, q) \in [0, \lambda] \times [0, 1]$, $\Omega(c_0, q) \neq 0$.

Proof. See the appendix. ■

Since $\Omega(c_0, q)$ is continuous in (c_0, q), the sufficient condition listed in Proposition 2 is equivalent to the requirement that $\Omega(c_0, q)$ does not change sign over $[0, \lambda] \times [0, 1]$.

[10] By Hölder's inequality, $\left[\int_0^1 \xi h(\phi, \xi) d\xi\right]^2 \leq \int_0^1 \xi^2 h(\phi, \xi) d\xi \int_0^1 h(\phi, \xi) d\xi$ (which never binds in this model). Thus $\Omega(c_0, q) \leq 1 + \int_0^1 \xi h(\phi(c_0, q, \xi), \xi) d\xi$.

Note that when either the reseller or the buyer has the full bargaining power ($\lambda = 1$ or $\lambda = 0$), it is easily seen that $\Omega(c_0, q) > 0$ for all $(c_0, q) \in [0, \lambda] \times [0, 1]$ and hence the symmetric entry equilibrium is unique (given any joint distribution H). By continuity, the uniqueness is also guaranteed when λ is sufficiently close to 1 or 0.

The condition identified by the above proposition is not the only sufficient condition for the uniqueness of equilibrium. In fact we can make the sufficient condition somewhat stronger, so that it will be easier to verify. One such stronger condition is developed below.

Note that $1 + \int_0^1 \xi h d\xi + \lambda(1-\lambda) \left[\int_0^1 \xi h d\xi\right]^2 \geq \left[2\sqrt{\lambda(1-\lambda)} + 1\right] \int_0^1 \xi h d\xi$, and that $\int_0^1 \xi^2 h d\xi < \int_0^1 \xi h d\xi$. So when $\max_{(c_0,q)\in[0,\lambda]\times[0,1]} \int_0^1 h(\phi(c_0, q, \xi), \xi) d\xi \leq \frac{2\sqrt{\lambda(1-\lambda)}+1}{\lambda(1-\lambda)}$, where $\lambda \in (0, 1)$, we have $\min_{(c_0,q)\in[0,\lambda]\times[0,1]} \Omega(c_0, q) > 0$, which guarantees the uniqueness of equilibrium. Note that for any given ξ, $h(\eta, \xi)$ is continuous in η over $[0, 1]$. Hence it achieves its maximum at some point, say, $\eta = m(\xi) \in [0, 1]$ (there may be multiple maximizers). An alternative (and somewhat stronger) sufficient condition for the uniqueness of equilibrium is thus given by:

$$\int_0^1 h(m(\xi), \xi) d\xi \leq \frac{2\sqrt{\lambda(1-\lambda)} + 1}{\lambda(1-\lambda)}, \tag{9}$$

where $m(\xi)$ maximizes $h(\eta, \xi)$ given ξ.

If there are multiple equilibria, it can be shown that the schedules of every two equilibrium entry cutoff curves cross exactly once (when plotted in the same graph). More specifically, suppose $C^I(v)$ and $C^{II}(v)$ represent two equilibria. Then there is a unique $v_0 \in (0, 1)$, such that $C^I(v) < C^{II}(v)$ when $v < v_0$, and $C^I(v) > C^{II}(v)$ when $v > v_0$. To show this, without loss of generality we assume that $C_0^I < C_0^{II}$. Then following a result established in the proof of Proposition 2, we have $q^I < q^{II}$. By (5), this in turn implies $C'^I(v) > C'^{II}(v)$, which means that although $C^I(v)$ starts below $C^{II}(v)$, it is everywhere steeper than $C^{II}(v)$. By (7), we cannot have $C^I(v) < C^{II}(v)$ for all v; otherwise $C_0^I < C_0^{II}$ cannot be true. Therefore, it must be the case that the two equilibrium schedules cross exactly once.

3.3 Resale vs. No Resale: A Comparison

Given any continuous and differentiable joint distribution function, $H(c, v)$, in general there are no closed-form solutions to the equilibrium entry cutoff curves $C(\cdot)$ and $\tilde{C}(\cdot)$; nevertheless we can establish the following comparison results.

Proposition 3 *For any* $\lambda \in (0,1]$, $C(v) > \tilde{C}(v)$ *for* v *sufficiently close to 0 and* $C(v) < \tilde{C}(v)$ *for* v *sufficiently close to 1. Therefore, the entry indifference curve* $\tilde{C}(\cdot)$ *crosses the entry indifference curve* $C(\cdot)$ *at least once over the range* $v \in (0,1)$.

Proof. See the appendix. ■

In the proof, we show that $C(0) > \tilde{C}(0)$ and $C(1) < \tilde{C}(1)$ for any $\lambda \in (0,1]$. So the proposition follows from the continuity of the two equilibrium entry cutoff curves. Thus those bidders with sufficiently low values can be referred to as *entry speculators*, as they enter the auction only because the resale opportunity is available. Those with sufficiently large values can be referred to as *bargain hunters*, as they stay out only because the resale opportunity is available and they might be able to strike a deal in the post-auction bargaining.

For $\lambda = 0$, it can be verified that $C(0) = \tilde{C}(0)$ and $C(1) < \tilde{C}(1)$. So the conclusion in Proposition 3 may not follow, which is intuitive: when the reseller does not have any bargaining power, the incentive for entry speculation is zero.

Note that we can express $\tilde{C}'(v) - C'(v)$ as follows:

$$\tilde{C}'(v) - C'(v) = (1-\lambda)(q-\tilde{q}) + \left[\lambda \int_v^1 \int_{\tilde{C}(\xi)}^1 + (1-\lambda) \int_0^v \int_0^{\tilde{C}(\xi)}\right] h(\eta,\xi) d\eta d\xi. \tag{10}$$

Thus when $\lambda = 1$, $\tilde{C}'(v) > C'(v)$ for all v, implying that $\tilde{C}(\cdot)$ crosses $C(\cdot)$ exactly once (by Proposition 3). By continuity, when λ is sufficiently close to 1, i.e., when the reseller has sufficiently large bargaining power, the types of entry speculators and bargain hunters are nicely separated by a single value threshold.

When $\lambda \in (0,1)$, it is worth noting that there is a connection between the probability of entry and the single crossing of the two entry indifference curves: from equation (10), it can be seen that as long as $q \geq \tilde{q}$ (namely, when there is more entry with resale), $\tilde{C}(\cdot)$ crosses $C(\cdot)$ exactly once.

4 Examples

To further understand the comparison, we follow a specific distribution family known as the Farlie-Morgenstern family, which is a special case of the copula:[11]

[11] A copula is a joint distribution function of random variables that have uniform marginal distributions.

$$H(c,v)=F(c)G(v)\{1+\alpha[1-F(c)][1-G(v)]\}$$

It can be easily verified that if F and G are both $U[0,1]$, then the marginal distributions generated by the F-M family are uniform distributions over $[0,1]$. In that case,

$$\begin{aligned}
H(c,v) &= (1+\alpha)cv-\alpha c^2v-\alpha cv^2+\alpha c^2v^2, \\
h(c,v) &= (1+\alpha)-2\alpha c-2\alpha v+4\alpha cv, \text{ and} \\
Corr(c,v) &= \frac{Cov(c,v)}{\sqrt{Var(c)Var(v)}}=\frac{\alpha}{3}.
\end{aligned} \tag{11}$$

Thus, the parameter α captures the correlation between the cost and value.

First of all, we will verify that this family satisfies the sufficient condition (9), hence the symmetric entry equilibrium in the case with resale is unique.

When $\lambda=0,1$, any joint distribution induces a unique equilibrium; when $\lambda\in(0,1)$ and $\alpha\geq 0$, it can be easily verified that $m(v)=0$ for $v\in[0,\frac{1}{2})$, and $m(v)=1$ for $v\in[\frac{1}{2},1]$, thus we have $\int_0^1 h(m(v),v)dv=1+\frac{\alpha}{2}$. Since $\min\frac{2\sqrt{\lambda(1-\lambda)}+1}{\lambda(1-\lambda)}=8$, $1+\frac{\alpha}{2}<\frac{2\sqrt{\lambda(1-\lambda)}+1}{\lambda(1-\lambda)}$. Similarly, when $\alpha<0$, $\int_0^1 h(m(v),v)dv=1-\frac{\alpha}{2}<\frac{2\sqrt{\lambda(1-\lambda)}+1}{\lambda(1-\lambda)}$. Therefore, the F-M family induces a unique equilibrium entry cutoff $C(\cdot)$ for any $\lambda\in[0,1]$.

When $\alpha=0$, c and v independently follow uniform distributions. Since it represents perhaps the simplest benchmark, we examine this case first.

For the no resale scenario, the differential equation system can now be simplified into the following system:

$$\begin{cases}\tilde{C}''(v)=\tilde{C}(v)\\ \tilde{C}(0)=0\\ \tilde{C}'(1)=1\end{cases}$$

From $\tilde{C}''(v)=\tilde{C}(v)$, we have $\tilde{C}(v)=Ae^v+Be^{-v}$ and $\tilde{C}'(v)=Ae^v-Be^{-v}$.

Making use of the two boundary conditions, we can solve the equilibrium entry cutoff curve:

$$\tilde{C}(v)=\frac{e}{e^2+1}(e^v-e^{-v}).$$

For the resale case, the equilibrium can also be solved explicitly, which is given by:

$$C(v) = c_0 + (1-\lambda)(1-q)v + \frac{\lambda v^2}{2}, \text{ where}$$

$$c_0 = \frac{\lambda(\lambda^2 - 7\lambda + 60)}{12(\lambda^2 - \lambda + 18)}, \text{ and } q = \frac{\lambda^2 + 2\lambda + 12}{2(\lambda^2 - \lambda + 18)}.$$

It can be easily verified that both c_0 and q increase in λ; namely, $c_0'(\lambda) > 0, q'(\lambda) > 0$. Thus in this case, when the reseller's bargaining power increases, entry becomes more attractive, entry probability increases, and it is also more likely for the lowest possible bidder (in terms of the value type) to enter the auction.

It turns out that Proposition 3 can be made more precise for this benchmark case with independent and uniform distributions.

Proposition 4 *When c and v are independently and uniformly distributed over $[0,1] \times [0,1]$, for any $\lambda \in (0,1]$, there exists a cutoff value $v^*(\lambda) \in (0,1)$, such that $C(v) > \tilde{C}(v)$ when $v \in (0, v^*(\lambda))$, and $C(v) < \tilde{C}(v)$ when $v > v^*(\lambda)$.*

Proof. See the appendix. ■

When λ is sufficiently large ($\lambda \geqslant \lambda^*$, for some $\lambda^* \in (0,1)$), we can show that $q(\lambda) \geqslant \widetilde{q}$. By appealing to (10), Proposition 4 follows immediately. When $\lambda \in (0, \lambda^*]$, however, the argument becomes quite tedious and the detailed proof is relegated to the appendix.

The comparison of the symmetric equilibrium entry cutoff curves is plotted below. For the case with resale, we plot the equilibrium entry cutoff curves for $\lambda = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

From the above figure, the cutoff value v^* increases in λ. So in this case, as the reseller's bargaining power increases, the (value) range over which a bidder enters the auction only because the opportunity of resale is available (speculative entry) becomes larger.

Now coming back to the case with the general correlation parameter α, we can derive the differential equation system characterizing the equilibrium for the case without resale, which is given by:

$$\begin{cases} \tilde{C}''(v) = (1 + \alpha - 2\alpha v)\tilde{C}(v) + (2\alpha v - \alpha)\tilde{C}^2(v) \\ \tilde{C}(0) = 0 \\ \tilde{C}'(1) = 1 \end{cases} \tag{12}$$

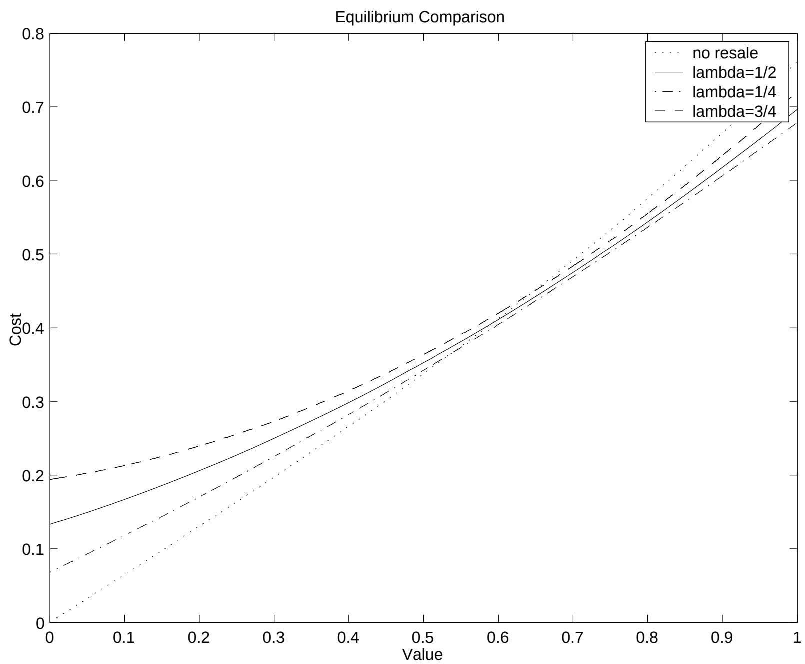

Figure 1:

For the case with resale, the equilibrium entry indifference curve can be solved explicitly, which is given by:

$$\begin{aligned} C(v) &= c_0 + (1-\lambda)(1-q)v + \lambda\frac{v^2}{2}, \text{ where} \qquad (13) \\ c_0 &= \lambda\int_0^1\int_{C(\xi)}^1 \xi h(\eta,\xi)d\eta d\xi, \text{ and } q = \int_0^1\int_0^{C(\xi)} h(\eta,\xi)d\eta d\xi \end{aligned}$$

where h is given in equation (11).

Based on the equilibrium entry cutoff curves, we compare the expected efficiency, ES (the expected total surplus generated from the sale, taking into account the entry costs incurred), and expected revenue (to the original item owner), ER, under both cases.

For the case with $\alpha = 0$, if we express the expected revenue and efficiency as functions of λ, they are given by:

$$ER^R = \frac{\lambda^6 - 44\lambda^5 + 6043\lambda^4 + 2568\lambda^3 + 121032\lambda^2 + 36288\lambda + 580608}{30240(\lambda^2 - \lambda + 18)^2}, \text{ and}$$
$$ES^R = \frac{-41\lambda^6 + 124\lambda^5 - 719\lambda^4 + 2148\lambda^3 + 129600\lambda^2 - 42336\lambda + 2576448}{30240(\lambda^2 - \lambda + 18)^2}.$$

It can be verified that ER^R is increasing in λ; thus as the reseller's bargaining power increases, expected revenue also increases. It can be verified that ES^R is concave in λ. Therefore, $ES^R(\lambda) \geq \min\{ES^R(0), ES^R(1)\}$. Based on this, it can be shown that resale is always more efficient than the no resale benchmark.

When comparing with the no resale benchmark case, it can be verified that expected revenue generated is lower when $\lambda \leq \lambda^*$, where $\lambda^* \approx 0.5856$.

From (12) and (13), we can analogously compute the entry probability, expected revenue, and expected surplus given different values of α. The following table reports the results given 5 correlation parameter values ($\alpha = 1$, $1/2$, 0, $-1/2$, -1), and 5 relative bargaining power parameter values ($\lambda = 1$, $3/4$, $1/2$, $1/4$, $\lambda = 0$).

	α	$\tilde{q}$	q	ER^{NR}	ER^{R}	ES^{NR}	ES^{R}
	1	0.335	0.4373	0.0538	0.0654	0.1857	0.2433
	$\frac{1}{2}$	0.343	0.4263	0.0603	0.0704	0.2123	0.2574
$\lambda = 1$	0	0.352	0.4167	0.0676	0.0762	0.2384	0.2720
	$-\frac{1}{2}$	0.362	0.4087	0.0757	0.0828	0.2499	0.2871
	-1	0.372	0.4024	0.0848	0.0902	0.2879	0.3026
	1	0.335	0.4050	0.0538	0.0580	0.1857	0.2454
	$\frac{1}{2}$	0.343	0.3992	0.0603	0.0640	0.2123	0.2588
$\lambda = \frac{3}{4}$	0	0.352	0.3947	0.0676	0.0708	0.2384	0.2729
	$-\frac{1}{2}$	0.362	0.3917	0.0757	0.0783	0.2499	0.2876
	-1	0.372	0.3903	0.0848	0.0867	0.2879	0.3029
	1	0.335	0.3721	0.0538	0.0518	0.1857	0.2431
	$\frac{1}{2}$	0.343	0.3722	0.0603	0.0585	0.2123	0.2571
$\lambda = \frac{1}{2}$	0	0.352	0.3732	0.0676	0.0661	0.2384	0.2716
	$-\frac{1}{2}$	0.362	0.3755	0.0757	0.0744	0.2499	0.2868
	-1	0.372	0.3790	0.0848	0.0836	0.2879	0.3025
	1	0.335	0.3401	0.0538	0.0470	0.1857	0.2364
	$\frac{1}{2}$	0.343	0.3460	0.0603	0.0542	0.2123	0.2521
$\lambda = \frac{1}{4}$	0	0.352	0.3526	0.0676	0.0623	0.2384	0.2683
	$-\frac{1}{2}$	0.362	0.3601	0.0757	0.0711	0.2499	0.2848
	-1	0.372	0.3685	0.0848	0.0808	0.2879	0.3015
	1	0.335	0.3096	0.0538	0.0437	0.1857	0.2251
	$\frac{1}{2}$	0.343	0.3212	0.0603	0.0511	0.2123	0.2441
$\lambda = 0$	0	0.352	0.3333	0.0676	0.0593	0.2384	0.2630
	$-\frac{1}{2}$	0.362	0.3459	0.0757	0.0683	0.2499	0.2816
	-1	0.372	0.3589	0.0848	0.0782	0.2879	0.3001

Some relatively clear patterns follow from the table. First, given α, in the case with resale, expected entry (q) and expected revenue (ER^R) both increase in λ, while expected surplus (ES^R) first increases, then decreases in λ when λ becomes sufficiently high. The intuition seems to be clear: a higher λ means more potential gain for the auction winner, which induces more entry. Higher entry leads to higher expected revenue; thus the expected revenue also increases in λ. However, when λ is sufficiently high, entry becomes excessive compared to the social optimum; thus expected surplus decreases as λ increases from $\frac{3}{4}$ to 1 (for all α). This suggests that the expected efficiency is concave in λ, as showed for the benchmark case $\alpha = 0$ above. Note that as λ increases from $\frac{3}{4}$ to 1, the magnitude by which ES^R decreases becomes smaller as α decreases. The intuition is that as α decreases, a bidder with a low value is more likely to incur a high entry cost, thus the excessive entry disappears gradually.

Given λ, for the no-resale scenario, $\tilde{q}$ decreases in α. This is intuitive as a lower α implies that given v, the associated entry cost is more likely to be lower, which makes entry more likely. In the resale scenario, q increases in α when λ is high ($\frac{3}{4}$ and 1) and decreases in α when λ is low ($0, \frac{1}{4}$ and $\frac{1}{2}$). Intuitively, this has to do with the two effects that a change in α may have on entry. When α increases, a bidder with a higher value is more likely to incur a higher entry cost thus has more incentive to stay out, while a bidder with a lower value is more likely to have a lower entry cost thus has more incentive to enter. The latter effect dominates the former effect when λ is high, exactly because this is when the incentive for speculative entry is the strongest. When λ is low, the opposite occurs and hence higher λ leads to lower entry.

When λ is fixed, both expected revenue and expected efficiency are decreasing in α in both the resale and non-resale cases. In the no-resale case, that α being lower means that an entrant is more likely to have a lower c and a higher v, which more likely contributes to higher expected surplus and expected revenue. In the resale case, a lower α corresponds to a lower possibility of speculative entry and bidders with higher values are more likely to enter the auction, leading to higher expected surplus and expected revenue.

By comparing the resale and non-resale cases, several observations are also in order. First, for all the cases reported, efficiency is always higher when resale is allowed. This is consistent with our initial intuition that allowing resale should help correct inefficiency caused by entry. However, the comparisons of expected entry and expected revenue depend on the values of the bargaining parameter λ. With resale, the expected entry is higher when λ is high ($\frac{1}{2}, \frac{3}{4}$and 1), ambiguous when $\lambda = \frac{1}{4}$, and lower when $\lambda = 0$). For the expected revenue, it is higher when $\lambda = \frac{3}{4}, 1$, and lower when $\lambda = 0, \frac{1}{4}, \frac{1}{2}$.

So while resale always increases efficiency, it reduces expected revenue when λ is not sufficiently large (e.g., $\lambda = 0, \frac{1}{4}, \frac{1}{2}$). This is not surprising in our model with only two bidders: while allowing resale may potentially increase competition in bidding and push up the revenue, this effect is absent here as whenever both bidders enter the auction, they know that the auction outcome is efficient and there will not be a post-auction resale. It is an open question whether our revenue comparison carries over to the setting with more than two bidders.

5 Concluding Remarks

In this work we explicitly take into account the possibility of post-auction resale in an auction model with entry where bidders possess two-dimensional private information. We demonstrate that the symmetric entry equilibrium is characterized by an entry cutoff curve, just as in the case when resale is absent, and we identify conditions under which such an equilibrium is unique. Our comparison results suggest that resale makes entry more likely for a bidder with a low value, and less likely for a bidder with a high value.

By following a specific distribution family which allows for any degree of correlation between entry costs and values, we can either analytically derive or numerically compute the equilibria in both situations when resale is allowed and not allowed. Our comparative static results suggest that allowing resale always increases the expected efficiency, which is consistent with our general intuition about the role resale can play in auctions with costly entry. So allowing resale is always beneficial from the social point of view. However, the effects of allowing resale on expected entry and expected revenue are ambiguous. For example, when the reseller has very low bargaining power, allowing resale reduces both entry and expected revenue; but when the reseller has sufficiently large bargaining power, allowing resale increases both entry and expected revenue. This finding suggests that the effects of allowing resale can be quite subtle on entry and expected revenue, suggesting that a seller's incentive to allow resale or not (if he can so choose) is completely nontrivial.

We have focused on second-price auctions in our analysis. But our results should not be altered if alternative auction formats are considered. This is due to revenue equivalence: as long as the resale mechanism is fixed, any standard auction (first-price, second-price, or all-pay auction) will induce exactly the same entry cutoff curve, which will in turn result in the same set of entrants.

One main restriction in our model is that we assume complete information in the resale stage (so that we can employ the Nash bargaining solution). Note that this assumption is perhaps less restrictive than it appears if we take into account the fact that, in our model with entry, the reseller behaves as if she knows quite a lot about the potential buyer: not only is her value higher than that of the reseller's, her value must also lie above the equilibrium entry cutoff curve (so the updated distribution of the buyer's value is much sharper in the resale stage). This being said, we assume complete information in the resale stage mainly for tractability of analysis; otherwise even when the post-auction resale takes the form of the simple take-it-or-leave it offer, we will encounter enormous complexity in deriving the differential equation system to characterize the equilibrium (for example, the boundary condition will be endogenously determined). Also note that given our assumption, there is no efficiency loss in our resale stage, which may potentially bias our efficiency comparison.[12] Despite all the restrictions, we demonstrate that our main results hold for any bargaining parameter $\lambda \in (0, 1]$. Thus as long as there is some potential gain from resale to the reseller, our main insights, though based on the complete information setup in the resale stage, should be fairly robust.

The other restriction in our analysis is that we only consider two bidders. As mentioned in the text, this simplification abstracts away an important effect of resale on bidding. Thus in a sense our current framework controls for the effect of resale on bidding and allows us to focus on its effect on entry only. To understand the compound effect of resale, however, extending our analysis to the case with more than two bidders is desirable. It turns out that with $n > 2$, however, it is hard to characterize equilibrium using an ODE system.[13] Despite all the technical difficulties, future research should extend our current analysis to a more general setting.

[12] It is not clear, though, whether our resale mechanism also leads to higher entry efficiency level through the induced entry cutoff curve.

[13] More specifically, we can no longer derive the form $C'(v) = g(C^{(0)}(v), ..., C^{(k-1)}(v), v)$ as in the case $n = 2$ to characterize entry equilibrium.

Appendix

Proof of Proposition 2:

Note that by (7), $c_0 = 0$ only when $\lambda = 0$, in which case $C(v)$ is linear, and the uniqueness of q_0 is easily established. Also by (7), $c_0 < 1$. Therefore, $c_0 \in (0,1)$ when $\lambda > 0$.

By (3), $q \in (0,1)$ since $C(v) \neq 0$, 1. Therefore, when $\lambda > 0$, $(c_0, q) \in (0,1) \times (0,1)$.

Next, from $\partial\gamma(q, c_0)/\partial q = -(1-\lambda)\int_0^1 \xi h(\phi(c_0, q, \xi), \xi)d\xi - 1 < 0$ for any c_0, we conclude that q_0 is unique for any given c_0. Moreover, by the implicit function theorem, in the open set $(0,1) \times (0,1)$, there exists a continuously differentiable function $g(\cdot)$ such that $q_0 = g(c_0)$.

It is easily verified that $g(\cdot)$ is strictly increasing in c_0 (the expression of $g'(c_0)$ is derived below).

Plug $q = g(c_0)$ back into (6), then $C(v)$ can be expressed in terms of c_0 and v. We will sometimes write $C(v)$ as $C(v, c_0)$ to emphasize its dependence on c_0:

$$C(v, c_0) = c_0 + (1-\lambda)(1-g(c_0))v + \lambda \int_0^v H(1,t)dt \tag{14}$$

After plugging (14) into (7), (7) becomes an equation involving c_0 only. To demonstrate that the solution of c_0 exists, we define:

$$S(c_0) = \lambda \int_0^1 \int_{C(\xi,c_0)}^1 \xi h(\eta, \xi)d\eta d\xi - c_0$$

Note that

$$\lambda \int_0^1 \int_{C(\xi,c_0)}^1 \xi h(\eta, \xi)d\eta d\xi \in (0, \lambda) \text{ for } \lambda > 0.$$

It is clear that for any $\lambda > 0$,we have $S(0) > 0$, and $S(1) < 0$. Thus there exists at least one $c_0 \in (0,1)$ such that $S(c_0) = 0$ (when $\lambda = 0$, $c_0 = 0$ is the solution). Define the set $\Gamma = \{c_0 \in [0,1] : S(c_0) = 0\}$, then $\Gamma \neq \varnothing$.

Therefore, the existence of $C(v)$ is guaranteed. Based on the arguments preceding Proposition 2, we conclude that there exists at least one symmetric entry equilibrium when resale is available.

We next identify sufficient conditions under which the equilibrium is unique. Equivalently, we identify sufficient conditions under which Γ is singleton.

One sufficient condition for Γ to be singleton is that $S'(c_0) > 0$ whenever $S(c_0) = 0$ or $S'(c_0) < 0$

whenever $S(c_0) = 0$ (by the single crossing lemma)[14]. Intuitively, if the derivatives evaluated at the crossing points are positive (or negative), then there is single crossing since $S(c_0)$ is continuously differentiable. We evaluate $S'(c_0)$ next.

$$\begin{aligned} S'(c_0) &= -\lambda \int_0^1 \xi h(C(\xi, c_0), \xi)[1 - (1-\lambda) g'(c_0)\xi] d\xi - 1 \\ &= -\lambda \int_0^1 \xi h(C(\xi, c_0), \xi) d\xi + \lambda(1-\lambda) g'(c_0) \int_0^1 \xi^2 h(C(\xi, c_0), \xi) d\xi - 1 \end{aligned}$$

where

$$g'(c_0) = \frac{\int_0^1 h(C(\xi, c_0), \xi) d\xi}{1 + (1-\lambda) \int_0^1 \xi h(C(\xi, c_0), \xi) d\xi}.$$

Therefore,

$$\begin{aligned} S'(c_0) &= -\frac{1}{1 + (1-\lambda) \int_0^1 \xi h(C(\xi, c_0), \xi) d\xi} \times \\ & \left[\begin{array}{c} 1 + \int_0^1 \xi h(C(\xi, c_0), \xi) d\xi + \lambda(1-\lambda) \left[\int_0^1 \xi h(C(\xi, c_0), \xi) d\xi \right]^2 \\ -\lambda(1-\lambda) \int_0^1 \xi^2 h(C(\xi, c_0), \xi) d\xi \int_0^1 h(C(\xi, a), \xi) d\xi \end{array} \right]. \end{aligned}$$

Clearly, one sufficient condition for the uniqueness of the equilibrium is that the numerator in the above expression is either strictly positive or strictly negative for any $c_0 \in \Gamma$.[15]

We can thus state a sufficient condition for the uniqueness of the symmetric equilibrium based on the function $\Omega(c_0, q)$ defined by (8) in the text.

By (7), $c_0 \in [0, \lambda) \subseteq [0, \lambda]$, hence $(c_0, q) \in \Gamma \times g(\Gamma) \subseteq [0, \lambda] \times [0, 1]$. The equilibrium is unique if for all $(c_0, q) \in [0, \lambda] \times [0, 1]$, $\Omega(c_0, q) \neq 0$ (which implies that either $\Omega(c_0, q) > 0$ or $\Omega(c_0, q) < 0$, by the continuity of Ω). ■

Proof of Proposition 3:

[14] Page 124, Milgrom (2004).

[15] Alternatively, we can combine (6) and (7) to get $c_0 = f(q)$, and obtain $C(v, q)$ by using (6). We can then plug it into (3) to get q. This alternative approach leads to exactly the same form for the numerator because $C(v, c_0) = C(v, f^{-1}(c_0)) = C(v, q)$.

It is easily verified that for any $\lambda > 0$, $C(0) > 0 = \tilde{C}(0)$. Hence by the continuity of $\tilde{C}(\cdot)$ and $C(\cdot)$, when $\lambda > 0$, $C(v) > \tilde{C}(v)$ for v sufficiently close to zero. We next show that $C(1) < \tilde{C}(1)$. Suppose in negation, $C(1) \geq \tilde{C}(1)$.

We will first argue that it cannot be the case that $C(v) \geq \tilde{C}(v)$ for all $v \in [0,1]$; otherwise we can argue that if $\tilde{C}(\cdot)$ characterizes the entry equilibrium under the regime without resale, then $C(\cdot)$ cannot characterize the entry equilibrium under the regime with resale. To see that, we can compare the entry incentive of a bidder with type $(C(1), 1)$ in the game with resale and the entry incentive of a bidder with type $(\tilde{C}(1), 1)$ in the game without resale. If $C(\cdot)$ and $\tilde{C}(\cdot)$ characterize entry equilibria under both regimes, then both types $(C(1), 1)$ and $(\tilde{C}(1), 1)$ should be indifferent between entry and staying out, given that the other bidder follows the proposed equilibria $C(\cdot)$ and $\tilde{C}(\cdot)$, respectively. This means that the expected net gain from entry over staying out for type $(C(1), 1)$, which is $C(1)$, is not lower than that for type $(\tilde{C}(1), 1)$, which is $\tilde{C}(1)$. But this is impossible: while both types $(C(1), 1)$ and $(\tilde{C}(1), 1)$ will win for sure upon entry (in their respective games), the expected payment conditional on winning for type $(C(1), 1)$ must be higher than that for type $(\tilde{C}(1), 1)$ (since $C(v) \geq \tilde{C}(v)$); on the other hand, the expected gain for type $(C(1), 1)$ from staying out is strictly positive given that resale occurs with strictly positive probability. Consequently, the expected net gain from entry (over staying out) for type $(C(1), 1)$ should be strictly lower than $\tilde{C}(1)$, the expected net gain from entry (over staying out) for type $(\tilde{C}(1), 1)$, a contradiction.

Therefore $\exists \hat{v} \in [0, 1)$, such that $C(\hat{v}) < \tilde{C}(\hat{v})$. Since by assumption, $C(1) \geq \tilde{C}(1)$, there must exist at least one crossing between $\hat{v}$ and 1. Let v' be the last crossing. This means that $C(v') = \tilde{C}(v')$ and $C(v) > \tilde{C}(v), \forall v \in (v', 1]$.

Note that for any $v \in [0, 1]$,

$$
\begin{aligned}
\tilde{C}'(v) - C'(v) &= \int_0^v \int_0^{\tilde{C}(\xi)} h(\eta, \xi) d\eta d\xi + \int_0^1 \int_{\tilde{C}(\xi)}^1 h(\eta, \xi) d\eta d\xi - \lambda \int_0^v \int_0^1 h(\eta, \xi) d\eta d\xi \\
&\quad -(1-\lambda) \int_0^1 \int_{C(\xi)}^1 h(\eta, \xi) d\eta d\xi \\
&= \left[(1-\lambda) \int_0^v \int_0^{C(\xi)} + \lambda \int_v^1 \int_{\tilde{C}(\xi)}^1 + (1-\lambda) \int_v^1 \int_{\tilde{C}(\xi)}^{C(\xi)} \right] h(\eta, \xi) d\eta d\xi.
\end{aligned}
$$

Because the sum of the first two integrals is positive, we have:

$$\tilde{C}'(v) - C'(v) > (1-\lambda) \int_v^1 \int_{\tilde{C}(\xi)}^{C(\xi)} h(\eta, \xi) d\eta d\xi. \tag{15}$$

That $C(v) > \tilde{C}(v), \forall v \in (v', 1]$ thus implies that $\tilde{C}'(v') - C'(v') > (1-\lambda) \int_{v'}^1 \int_{\tilde{C}(\xi)}^{C(\xi)} h(\eta, \xi) d\eta d\xi \geqslant 0$, which is a contradiction because when v' is the last intersection point and $C(v) > \tilde{C}(v), \forall v \in (v', 1]$, we should have $\tilde{C}'(v') \le C'(v')$. The proposition follows from the continuity of the entry cutoff curves. ■

Proof of Proposition 4:

It can be verified that $q'(\lambda) > 0$. Since $\widetilde{q}$ is independent of λ, there exists a cutoff $\lambda^* \in (0, 1]$ such that $q(\lambda) \ge \widetilde{q}$ if and only if $\lambda \ge \lambda^*$ (our computation suggests that $\lambda^* \approx 0.2409$). By the arguments following equation (10), we conclude that when $\lambda \ge \lambda^*$, there must be a single crossing between two entry indifference curves.

It remains to argue that the crossing is unique for $\lambda \in (0, \lambda^*]$. Since $q(1/4) > \widetilde{q}$, $\lambda^* < 1/4$. So the rest of the proof it suffices to argue that the crossing is unique for $\lambda \in (0, 1/4]$.

Let $D(v) = \tilde{C}'(v) - C'(v)$. Then

$$\begin{aligned} D'(v) &= \frac{e}{e^2+1}(e^v - e^{-v}) - \lambda, \\ D''(v) &= \frac{e}{e^2+1}(e^v + e^{-v}) > 0. \end{aligned} \tag{16}$$

Since $D'(v)$ is increasing in v, $D'(0) < 0$, and $D'(1) > 0$ when $\lambda \in (0, 1/4]$, there is a unique $v^*(\lambda) \in (0, 1)$ such that $D'(v^*(\lambda)) = 0$. Therefore, $D(v)$ is convex, decreasing when $v < v^*(\lambda)$ and increasing when $v > v^*(\lambda)$. $v^*(\lambda)$ can be solved explicitly from equation (16), and it can be verified that $v^*(\lambda)$ is increasing in λ.

Proposition 3 establishes that there should be at least one crossing. If $D(v^*(\lambda)) > 0$, then $D(v) > 0$ for all $v \in [0, 1]$ and hence the crossing is unique. Note that

$$D(v^*(\lambda)) = \frac{e}{e^2+1}(e^{v^*(\lambda)} + e^{-v^*(\lambda)}) - \lambda v^*(\lambda) - (1-\lambda)(1-q(\lambda)).$$

It can be verified that $D(v^*(\lambda))$ is strictly increasing in λ for $\lambda \in (0, 1/4]$, and that $D(v^*(0)) < 0$ and $D(v^*(1/4)) > 0$. Thus there is a unique $\lambda^{**} \in (0, 1/4)$ such that $D(v^*(\lambda^{**})) = 0$ (our computation indicates that $\lambda^{**} \approx 0.027$). Therefore, when $\lambda \in (\lambda^{**}, 1/4]$, the crossing is unique.

It remains to show that the crossing is unique for $\lambda \in (0, \lambda^{**}]$.

We now consider $\lambda \in (0, \lambda^{**}]$. Since $\tilde{C}(0) < C(0)$ and $\tilde{C}(1) > C(1)$, at the first crossing point $v = v_1$, it must be the case that $\tilde{C}'(v_1) > C'(v_1)$, or $D(v_1) > 0$. Clearly, $v_1 > v^*(\lambda)$ leads to a unique crossing. Moreover, to argue that $v_1 > v^*(\lambda)$, it suffices to show $D(0) \leq 0$, where $D(0) = 1 - \tilde{q} - (1-\lambda)(1-q(\lambda))$.

Since $q'(\lambda) > 0$, $D(0)$ is increasing in λ. Since $D(0) < 0$ when $\lambda = 0$ and $D(0) > 0$ when $\lambda = \lambda^{**}$ (when $\lambda = \lambda^{**}$, $D(0) > D(v(\lambda^{**})) = 0$), there is a unique $\lambda^{***} \in (0, \lambda^{**})$ such that $D(0) = 0$ when $\lambda = \lambda^{***}$. Therefore, when $\lambda \in (0, \lambda^{***}]$, $D(0) \leq 0$ thus the crossing is unique (our computation shows that $\lambda^{***} \approx 0.0253$).

Now it remains to show that the crossing is unique when $\lambda \in (\lambda^{***}, \lambda^{**}]$, where $\lambda^{**} \approx 0.027 < 0.1$ (since it can be verified that $D(v(0.1)) > 0$). It thus suffices to show that for $\lambda \in (\lambda^{***}, 0.1]$, $v_1(\lambda) > v^*(\lambda)$. The rest of the proof is completed in the following steps:

1. It can be shown that $C(v)$ is increasing in λ (given v) for $(\lambda, v) \in (0, 0.1] \times [0, \tilde{v}]$ for some $\tilde{v} \in (0, 1]$. Hence for $0.1 \geqslant \lambda_1 > \lambda_2 > 0$, $v_1(\lambda_1) > v_1(\lambda_2)$ as long as $v_1(\lambda_2) \in [0, \tilde{v}]$. Therefore, for any $\lambda_2 < \lambda^{***}$ such that $v_1(\lambda_2) \in [0, \tilde{v}]$, $v_1(\lambda)$ is bounded below by $v_1(\lambda_2)$ for $\lambda \in (\lambda^{***}, 0.1]$.

2. For $\lambda \in (0, 0.1]$, $v^*(\lambda) < v^*(0.1)$.

3. If there exists a $\lambda_2 \in (0, \lambda^{***})$ such that $v_1(\lambda_2) \in [0, \tilde{v}]$, then as long as $v_1(\lambda_2) > v^*(0.1)$, for $\lambda \in (0, 0.1]$ we have $v_1(\lambda) > v_1(\lambda_2) > v^*(0.1) > v^*(\lambda)$.

   Thus the proof is completed once such a λ_2 exists. A simple calculation shows that $\tilde{v} > 0.6$.[16] Any number in the interval $(0, \lambda^{***})$ satisfies the requirement since $v_1(\lambda^{***})$ is less than 0.5.

   In summary, for any $\lambda \in (0, 1]$, the schedule of $C(v)$ crosses the schedule of $\tilde{C}(v)$ only once. ■

[16] We obtain $\tilde{v}$ by solving $\frac{\partial C(v,\lambda)}{\partial \lambda} > 0$ for $(\lambda, v) \in (0, 0.1] \times [0, \tilde{v}])$ and $v^*(0.1) < 0.16$. Any number in $(0, \lambda^{***})$ is a candidate for λ_2 because $v_1(\lambda_2)$ is sufficiently smaller than $\tilde{v}$ and larger than $v^*(0.1)$ for $\lambda_2 \in (0, \lambda^{***})$ (we have argued that there is a unique crossing for $\lambda \in (0, \lambda^{***})$ thus we can estimate the value).

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