

# Auctions with Entry and Resale\*

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## Abstract

We study how resale affects auctions with costly entry in a model where bidders possess two-dimensional private information signals: entry costs and valuations. We establish the existence of symmetric entry equilibrium and identify sufficient conditions under which the equilibrium is unique. Our analysis suggests that the opportunity of resale affects both entry and bidding, and, in particular, it induces motivation for both speculative entry and resale hunting abstentions. Our numerical results suggest that while the entry probability and efficiency are always higher when resale is allowed, the effect on the seller's expected revenue is ambiguous.

## 1 Introduction

Starting with the seminal work of Vickrey (1961) and continuing until recently, the auction literature has mostly adopted the paradigm of a fixed number of bidders. This simplifying assumption has

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allowed for enormous progress in the characterization of optimal auctions and analysis of revenue equivalence and revenue rankings of different auction formats (see, for example, Riley and Samuelson (1981), Myerson (1981), Milgrom and Weber (1982)). However, in many auctions the number of rivals is not known when bids are placed.<sup>1</sup> This observation motivated several papers (e.g., McAfee and McMillan (1987), Matthews (1987), and Harstad, Kagel, and Levin (1990)) to treat the number of rival bidders as coming from exogenously determined distributions. In models with such an extension, bidders may have preferences over different auction formats, which in turn has important impact on optimal auction designs. In addition, auctioneers may have one more design instrument regarding whether to reveal or to conceal the number of bidders.<sup>2</sup> However, this approach with a stochastic number of bidders still ignores an important issue. In many auctions the cost of bid preparation or information acquisition is far from trivial. We cannot simply rank auctions by assuming a fixed or stochastic number of bidders without accounting for the fact that different auctions are likely to induce different entry incentives. Endogenous entry must be taken into account in order to compare expected revenue or to design optimal auctions in those situations.

These considerations have motivated a growing literature on auctions with costly and endogenous entry.<sup>3</sup> Independent-private-value (IPV) auctions with costly entry often result in inefficient allocation, since the auction is typically conducted among the actual bidders, which is a subset of all potential bidders. If the bidder with the highest value is excluded from the auction, the auction outcome is necessarily inefficient *ex post*.

This sort of inefficiency may create a motive for post-auction resale and such resale opportunity affects significantly, as we shall show, bidders' bidding behavior and entry strategies. In this paper, we study the effect of allowing resale opportunity in an auction model where each potential bidder possesses two-dimensional private information signals: entry cost ( $c$ ) and valuation ( $v$ ). When the opportunity of resale is absent, this framework is first analyzed by Green and Laffont (1984), who

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<sup>1</sup>This is the case, for example, in most of the sealed-bid procurement auctions.

<sup>2</sup>See Harstad et al. for how to implement such a scheme even when the seller does not know *ex ante* the actual number of bidders. Also see Levin and Ozdenren (2004) who revisit these questions under ambiguity.

<sup>3</sup>See, among others, French and McCormick (1984), Green and Laffont (1984), Samuelson (1985), McAfee and McMillan (1987), Tan (1992), Engelbrecht-Wiggans (1993), Levin and Smith (1994), Stegeman (1996), Tan and Yilankaya (2006), Gal et al. (2007), Ye (2004, 2007), Lu (2006, 2007), and Moreno and Wooders (2011). Also see Bergemann and Välimäki (2006) for an extensive survey of the literature.

demonstrate that under a Vickrey auction the entry equilibrium is characterized by a unique entry cutoff curve (or entry indifference curve)  $C(\cdot)$  such that a bidder with type  $(c, v)$  enters the auction if and only if  $c \leq C(v)$ . In such an equilibrium, it is possible that a bidder with a high value chooses not to enter the auction, simply because her entry cost is above her entry cutoff. Following Green and Laffont's analysis, Lu (2006) characterizes the symmetric entry equilibrium cutoff curves implemented by the classes of *ex post* efficient or *ex post* revenue-maximizing mechanisms. In a general procurement setting allowing for correlations between participation costs and production costs, Gal, Landsberger, and Nemirovski (2007) establish the existence and uniqueness of the equilibrium entry cutoff curve.<sup>4</sup>

Introducing post-auction resale into models with costly entry enriches the analysis, yet complicates matters so that the existence and uniqueness of entry equilibrium are no longer obvious. The additional difficulty emerges since, with resale, the option value for staying out is also positive and varies with types. To make our analysis tractable, instead of working with continuous entry cost types, we assume that a bidder's entry cost can only be either high ( $c_H$ ) or low ( $c_L$ ) in this paper.<sup>5</sup> With such a modeling simplification, we are able to show that with resale, a symmetric entry equilibrium is characterized by entry thresholds  $(v_L, v_H)$ , where  $v_H > v_L$ , so that bidders with entry cost  $c_L$  (henceforth often referred to as type- $L$  bidders) will enter the auction if and only if their values  $v \geq v_L$ , and bidders with entry cost  $c_H$  (henceforth often referred to as type- $H$  bidders) will enter the auction if and only if their values  $v \geq v_H$ . We show that while the opportunity of resale only affects the entry threshold for type- $H$  bidders, it affects both the entry threshold and bidding for type- $L$  bidders. More specifically, type- $L$  bidders tend to bid more aggressively with the resale opportunity, and the lower the value, the higher the magnitude of overbidding above value.

While the symmetric entry equilibrium is unique in the no-resale benchmark, the uniqueness cannot be established when resale is allowed. In the case with multiple equilibria, we show that the equilibria can be nicely ranked in the increasing order of  $v_L$ , or equivalently, in the decreasing order

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<sup>4</sup>Moreover, they demonstrate that providing participation reimbursements can partially mitigate the problem with entry and improve the seller's expected revenue.

<sup>5</sup>Working with continuous entry costs should not alter the main insights of our analysis. However, as demonstrated in Xu et al. (2010), when a bidder's type  $(c, v)$  are continuous and drawn from some general distribution  $H(c, v)$ , the equilibrium analysis is only tractable when the number of potential bidders  $n = 2$  and when the resale is conducted under complete information.

of  $v_H$ .<sup>6</sup> We also identify sufficient conditions that assure the uniqueness of the symmetric entry equilibrium.

We compare the symmetric entry equilibrium when resale is allowed to the equilibrium when resale is banned. Our first finding is that with resale, the entry cutoff is higher for bidders with higher entry costs (type- $H$  bidders), and lower for bidders with lower entry costs (type- $L$  bidders). This suggests that when resale is allowed, bidders with low entry costs are more likely to enter, while bidders with high entry costs are less likely to enter. In other words, resale naturally induces a *speculative* motivation for entry and a *resale-hunting* motivation for staying out. Speculators are those with low entry costs and low valuations who would not enter without resale, but enter when resale opportunity is available. Resale hunters are those with high entry costs and high valuations who would refrain from entering when a resale market is available, but enter when resale is unavailable. Thus, although we may be tempted to think that the additional opportunity to trade in the resale market provides an additional motive to enter, the opposite incentive to avoid entry cost by staying out and buying in the resale market may more than offset it. Overall, the resale opportunity introduces both incentives for entry and staying out, and the net effect is ambiguous except for those with low entry costs and low valuations (entry speculators) and those with high entry costs and high valuations (resale hunters).

In our model, it is difficult to work out a closed-form solution for the equilibrium with a general value distribution. In order to get some ideas about how resale opportunities affect the expected entry, efficiency, and revenue, we follow a uniform distribution example. Our numerical results reveal the following patterns: the expected entry and efficiency is always higher when resale is allowed, which is consistent with our general notion that allowing resale should help correct the inefficiency induced by costly entry. However, the impact of resale on expected revenue is somewhat ambiguous, although for almost all the parameter values used for our numerical computation, the expected revenue is lower when resale is allowed.

The literature on auctions with resale is relatively new. Gupta and Lebrun (1999) consider first-price auctions with resale, where complete information is assumed in the resale stage. Haile (2003) considers an IPV setup in which bidders only have noisy signals at the auction stage, where the motive for resale arises when the true value of the auction winner turns out to be low. Zheng (2002)

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<sup>6</sup>In Section 5, we discuss how cheap talk can be used to coordinate entry in the presence of multiple equilibria.

identifies conditions under which the outcome of optimal auctions can be achieved with resale. Garret and Tröger (2006) consider a model with a speculator, who can only benefit from participating in the auction when she can resell the item to the other bidder. Pagnozzi (2007) demonstrates why a strong bidder may prefer to drop out of an auction before the price reaches her valuation, simply because she anticipates that she will be in an advantageous position in the post-auction resale stage.<sup>7</sup> Hafalir and Krishna (2008) analyze auctions with resale in an asymmetric IPV auction environment with two bidders and show that the expected revenue is higher under a first-price auction than it is in a second-price auction. Finally, Garratt, Tröger, and Zheng (2009) show that when resale is allowed, the English ascending auction is susceptible to tacit collusion. They construct equilibria that interim Pareto dominate the standard truthful value-bidding equilibrium. Our paper contributes to the literature by being the first to integrate and analyze both entry and resale in an auction model with two-dimensional private information, an arbitrary number of bidders, and an incomplete information setting for the resale stage.

The paper is organized as follows. Section 2 lays out the model. Section 3 characterizes the equilibrium and derives sufficient conditions under which the equilibrium is unique under both no-resale and resale cases. Section 4 is devoted to comparisons of the equilibria in the two cases, and provides some numerical results for a uniform distribution example. Section 5 discusses some assumptions and extensions in our analysis, and Section 6 concludes.

## 2 The Model

There is a single, indivisible object for sale to  $n \geq 2$  potentially interested buyers (firms) through an auction. The seller's valuation is normalized to 0. It is costly for a bidder to participate in the auction (e.g., it is costly to prepare blueprints, to hire experts for best bidding strategies, or simply to establish eligibility to bid or to meet some legal requirements, etc.). So, unlike most auction models, in our model each bidder possesses two-dimensional private information about her participation cost ( $c$ ) and value ( $v$ ). We assume that  $c = c_L$  with probability  $q$  and  $c = c_H$  with probability  $1 - q$ , and  $0 < c_L < c_H < 1$ ;<sup>8</sup>  $v$  is distributed according to the cumulative distribution function  $F(\cdot)$  on  $[0, 1]$ .

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<sup>7</sup>In Pagnozzi (2007), resale is conducted in a complete information setting.

<sup>8</sup>When  $c_L = c_H = c$ , resale never occurs in the symmetric monotone entry equilibrium we will focus on, hence is omitted in our analysis.

We assume that for any  $a \in (0, 1]$ , the truncated distribution of  $F(\cdot)$  on  $[0, a]$  satisfies the monotonic hazard rate property, that is,  $[F(a) - F(x)]/f(x)$  strictly decreases in  $x$  for  $x \in [0, a]$ . We assume that a second-price sealed-bid auction is conducted, which is without loss of generality as we will show that revenue equivalence holds among all the standard auctions in our setting. For simplicity of analysis, we assume that the seller does not set a reserve price other than his own reservation value 0. After the auction, the winner can conduct a post-auction resale.<sup>9</sup> In the resale, we assume that the winner of the item from the initial auction (the reseller) will conduct an optimal auction (with an optimal reserve price) to resell the item.

The seller makes the selling mechanism publicly known before participation (or entry) occurs. After learning the realizations of their private entry costs and values, the potential bidders make entry decisions simultaneously and independently. After entry, the bidders bid for the item. We will consider the two scenarios when resale is allowed and not allowed.

### 3 Equilibrium Characterizations

In this section we characterize the equilibrium in the no-resale benchmark case and in the case when resale is allowed. We will focus on the symmetric entry equilibrium characterized by a pair of entry thresholds,  $(\tilde{v}_L, \tilde{v}_H)$  in the case when resale is banned and  $(v_L, v_H)$  in the case when resale is allowed, so that a bidder with a type  $(c_k, v)$  enters the auction if and only if  $v \geq \tilde{v}_k$  in the no resale benchmark and  $v \geq v_k$  in the case with resale,  $k \in \{L, H\}$ ; in other words, bidders with the same entry cost type will follow the same entry threshold in a symmetric entry equilibrium.

#### 3.1 Entry without Resale

In this section we analyze the benchmark case when resale is banned. Since bidders are *ex ante* symmetric, without loss of generality we can focus on bidder  $i$ 's entry strategy. To show that  $(\tilde{v}_L, \tilde{v}_H)$  characterizes a symmetric entry equilibrium, we need to verify that there is no incentive for bidder  $i$  to deviate from the prescribed entry strategy given that all the other bidders follow the entry thresholds  $(\tilde{v}_L, \tilde{v}_H)$ .

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<sup>9</sup>There is no obvious reason to believe that the heterogenous entry costs that we envision also apply to the resale. We thus assume that the entry costs for the resale market are the same. We further assume that this cost is small, and is hence omitted. We discuss the implications of this assumption in Section 5.

First, given that  $0 < c_L < c_H < 1$ , we can rule out both  $\tilde{v}_L = 0$  and  $\tilde{v}_H = 0$  (otherwise bidders will incur net losses with realized values lower than their entry costs). We can also rule out the case  $\tilde{v}_L = \tilde{v}_H = 1$  (otherwise a single bidder with a value higher than her entry cost would find it profitable to deviate to enter the auction). We next show that in any symmetric entry equilibrium,  $0 < \tilde{v}_L < \tilde{v}_H \leq 1$ . Suppose in negation we have  $1 \geq \tilde{v}_L > \tilde{v}_H > 0$  or  $1 > \tilde{v}_L = \tilde{v}_H > 0$ . But then a bidder with type  $(c_L, \tilde{v}_H)$  who is supposed to stay out (or be indifferent between entry and staying out) would be strictly better off by entering, as a bidder with  $(c_H, \tilde{v}_H)$  is indifferent between entry and staying out, a contradiction. Therefore, given that  $0 < c_L < c_H < 1$ , it must be the case that  $0 < \tilde{v}_L < \tilde{v}_H \leq 1$  in any symmetric entry equilibrium.

When all the other bidders follow the pair of entry thresholds  $(\tilde{v}_L, \tilde{v}_H)$ , the ex ante probability that a bidder enters the auction is given by:

$$\tilde{p}_E = 1 - qF(\tilde{v}_L) - (1 - q)F(\tilde{v}_H).$$

Let  $\tilde{F}_{in}(x)$  be the probability that a bidder has a value less than  $x$  conditional on entry. Then  $\tilde{F}_{in}(x) = q[F(x) - F(\tilde{v}_L)]/\tilde{p}_E$  when  $x \in [\tilde{v}_L, \tilde{v}_H)$  and  $\tilde{F}_{in}(x) = [F(x) - qF(\tilde{v}_L) - (1 - q)F(\tilde{v}_H)]/\tilde{p}_E$  when  $x \in [\tilde{v}_H, 1]$ . Let  $\tilde{f}_{in}(\cdot)$  be its associated density function. Similarly, let  $\tilde{F}_{out}(x)$  be the probability that a bidder has a value less than  $x$  conditional on staying out. Then  $\tilde{F}_{out}(x) = F(x)/(1 - \tilde{p}_E)$  when  $x \in [0, \tilde{v}_L]$  and  $\tilde{F}_{out}(x) = [qF(\tilde{v}_L) + (1 - q)F(x)]/(1 - \tilde{p}_E)$  when  $x \in [\tilde{v}_L, \tilde{v}_H]$ . Let  $\tilde{f}_{out}(\cdot)$  be its associated density function.

We assume that each bidder plays the (weakly) dominant strategy to bid her value after entry.<sup>10</sup> We use  $\tilde{\pi}(v, m, \tilde{v}_L, \tilde{v}_H)$  to denote a bidder's expected payoff conditional on entry, given her value  $v$ , the total number of entrants  $m$ , and the pair of entry thresholds followed by all the other bidders  $(\tilde{v}_L, \tilde{v}_H)$ . Then

$$\tilde{\pi}(v, m, \tilde{v}_L, \tilde{v}_H) = \int_{\tilde{v}_L}^v (v - x)\tilde{f}_{in,m-1}^{(1)}(x)dx = \int_{\tilde{v}_L}^v \tilde{F}_{in,m-1}^{(1)}(x)dx,$$

where  $\tilde{f}_{in,m-1}^{(1)}(x)$  and  $\tilde{F}_{in,m-1}^{(1)}(x)dx$  are the density function and cumulative distribution function, respectively, for the highest value possessed by all the other  $m - 1$  entrants.

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<sup>10</sup>Note that participation costs are sunk costs, which should not affect the bidding strategies (absent the resale opportunity).

Given that everyone else follow the symmetric entry thresholds, the number of other entrants follows a binomial distribution; that is,  $\Pr(m-1 \text{ entrants out of } n-1 \text{ other bidders}) = C_{n-1}^{m-1}(\tilde{p}_E)^{m-1}(1-\tilde{p}_E)^{n-m}$ . Therefore, if we use  $\tilde{\pi}(v, \tilde{v}_L, \tilde{v}_H)$  to denote the expected total payoff conditional on entry, we have

$$\tilde{\pi}(v, \tilde{v}_L, \tilde{v}_H) = \sum_{m=1}^n C_{n-1}^{m-1}(\tilde{p}_E)^{m-1}(1-\tilde{p}_E)^{n-m}\tilde{\pi}(v, m, \tilde{v}_L, \tilde{v}_H).$$

It can be easily verified that given  $(\tilde{v}_L, \tilde{v}_H)$ ,  $\tilde{\pi}(v, \tilde{v}_L, \tilde{v}_H)$  strictly increases in  $v$ . Therefore, given that other bidders follow the entry thresholds  $(\tilde{v}_L, \tilde{v}_H)$ , no bidder has an incentive to deviate from following the same entry thresholds.

**Proposition 1** *In the benchmark case when resale is banned, there exists a unique symmetric entry equilibrium characterized by the entry thresholds  $(\tilde{v}_L, \tilde{v}_H)$ .*

**Proof.** See Appendix. ■

In particular, if the values of  $(c_L, c_H)$  induce a pair of interior entry thresholds  $(\tilde{v}_L, \tilde{v}_H)$ , i.e.,  $0 < \tilde{v}_L < \tilde{v}_H < 1$ , we must have the following indifference conditions:

$$\begin{cases} \tilde{\pi}(\tilde{v}_L, \tilde{v}_L, \tilde{v}_H) - c_L = 0 \\ \tilde{\pi}(\tilde{v}_H, \tilde{v}_L, \tilde{v}_H) - c_H = 0 \end{cases} \quad (1)$$

That is, a bidder with an entry cost  $c_L$  ( $c_H$ ) will be indifferent between entering the auction and staying out if her value is  $\tilde{v}_L$  ( $\tilde{v}_H$ ).<sup>11</sup>

Since the interior entry thresholds are the main focus in our analysis, we also fully characterize the set of all pairs of  $(c_L, c_H)$  that can induce a pair of interior entry thresholds.

**Proposition 2** *The set of all pairs of  $(c_L, c_H)$  that can induce a pair of interior entry thresholds,  $\tilde{Y} = \{(c_L, c_H) : c_L \in (0, 1), c_H \in (c_L, \alpha(c_L))\}$ , where  $\alpha(c_L)$  is the highest  $c_H$  that can, given  $c_L$ , induce a solution to the equation system (1).*

**Proof.** See Appendix. ■

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<sup>11</sup>If  $(c_L, c_H)$  cannot induce an interior solution to the equation system (1), it has to be the case that  $\tilde{v}_L \in (0, 1)$  and  $\tilde{v}_H = 1$ . Note that  $\tilde{v}_L = 0$  is impossible as we assume that  $c_L > 0$ .

### 3.2 Entry with Resale

We now augment the model just analyzed by allowing a resale stage where the auction winner may resell the item. Similarly to the benchmark case without resale, we will focus on symmetric entry equilibria characterized by entry thresholds  $(v_L, v_H)$ .

A bidder's ex ante probability of entry,  $p_E$ , the value distribution function conditional on entry,  $F_{in}(\cdot)$ , and the value distribution function conditional on staying out,  $F_{out}(\cdot)$ , are all defined analogously as in the no-resale case.

With resale opportunities, given that  $0 < c_L < c_H < 1$ , we can again rule out the case  $v_L = v_H = 1$ . However, unlike in the case without resale, now either  $v_L = 0$  or  $v_H = 0$  (but not both) is possible, given that the additional value from the opportunity of resale may more than offset the entry cost. Following a similar argument as in the no-resale case, we can conclude that in any symmetric entry equilibrium,  $0 \leq v_L < v_H \leq 1$ .

To characterize symmetric entry equilibria, we start our analysis from the last stage. Given the special features in our model, it is easily seen that in equilibrium, the initial auction winner can only possibly benefit from resale when her value  $v_w \in [v_L, v_H)$ . So resale can only be initiated when some type- $L$  bidder wins the initial auction, and the potential buyers participating in resale must be type- $H$  bidders with values in  $[v_L, v_H]$  who stay out of the initial auction.<sup>12</sup> When resale is initiated, the resale mechanism will be announced (we assume that the reseller uses an optimal auction *à la* Myerson); in particular, the reserve price will be announced. Following the announcement, the potential buyers will come to the resale and participate in bidding.

So, in our model, resale provides an opportunity for bidders with low value and low cost to speculate: they can sell the item to a bidder with higher values who also has higher entry cost. In the resale, the seller sets an optimal reserve price to maximize her expected payoff, with the belief that potential buyers' values follow the distribution  $F_{out}(\cdot)$ . In equilibrium, a trade in resale occurs only when the outside bidder with the highest value can meet the reserve price.

Define  $w(v, m, v_L, v_H)$  to be the reseller's expected gross payoff from resale, given her value  $v$ , the number of entrants (including herself)  $m$ , and the entry thresholds that other bidders follow  $(v_L,$

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<sup>12</sup>Note that resale will never be initiated in equilibrium whenever some bidder has a value higher than  $v_H$ , as the initial auction winner must possess the highest value among all the  $n$  potential buyers.

$v_H$ ). We have

$$w(v, m, v_L, v_H) = \max_{r \in (v, v_H)} [F_{out}(r)]^{n-m} v + C_{n-m}^1 [1 - F_{out}(r)] [F_{out}(r)]^{n-m-1} r + \int_r^{v_H} x f_{out, n-m}^{(2)}(x) dx. \quad (2)$$

The first term reflects the reseller's payoff when no bidder bids higher than the reserve price, the second term reflects the reseller's payoff when only one bidder bids higher than the reserve price, and the third term reflects the reseller's payoff when at least two bidders bid above the reserve price.

The optimal reserve price,  $r(v)$ , is the solution to equation  $x = v + (1 - F_{out}(x)) / f_{out}(x)$ . When the truncated distribution of  $F(\cdot)$  on  $[0, a]$ ,  $a \leq 1$ , satisfies the monotone hazard rate property, there is only one solution to this equation.

Next we consider a bidder (with a value  $v$ ) who stays out of the auction but participates in the resale. Such a bidder can only have a positive expected payoff when  $v > r(v_L)$ , as  $r(v_L)$  is the lowest possible reserve price in the resale. Let  $y$  be the highest value among the other  $n - m - 1$  bidders in the resale and  $z$  be the reseller's value (the highest value among  $m$  entrants). Then this outsider's expected payoff from resale is given by:

$$L(v, m, v_L, v_H) = \begin{cases} 0 & \text{if } v \leq r(v_L) \\ \int_{v_L}^{r^{-1}(v)} \int_{r(z)}^v (v - y) f_{out, n-m-1}^{(1)}(y) f_{in, m}^{(1)}(z) dy dz \\ + \int_{v_L}^{r^{-1}(v)} \left[ \int_0^{r(z)} f_{out, n-m-1}^{(1)}(y) dy \right] [v - r(z)] f_{in, m}^{(1)}(z) dz & \text{if } v > r(v_L) \end{cases}$$

Define

$$\beta(v, m, v_L, v_H) = \begin{cases} w(v, m, v_L, v_H) & \text{if } v \leq v_H \\ v & \text{if } v > v_H \end{cases} \quad (3)$$

Clearly,  $\beta(v, m, v_L, v_H)$  is the expected payoff (gross of entry cost) conditional on winning the initial auction, taking into account the additional value coming from potential resale (for those who will initiate a resale auction with positive probability).

**Proposition 3** *In the symmetric equilibrium when resale is allowed, entrant bidders follow the bid function  $\beta(\cdot, m, v_L, v_H)$  in the initial auction. Bidders bid above value for  $v < v_H$  and the magnitude of overbidding,  $\beta(v, m, v_L, v_H) - v$ , is decreasing in  $v$ .*

**Proof.** See Appendix. ■

Given the specific information structure in our model, resale will never be initiated by a type- $H$  bidder, so, not surprisingly, resale has no effect on bidding for type- $H$  bidders. However, as suggested in Proposition 3, resale affects bidding for type- $L$  bidders. This is reflected by overbidding above value for those with valuations between  $v_L$  and  $v_H$  (who will possibly initiate resale after the initial auction). For those bidders, Proposition 3 shows that the lower the value, the more the magnitude of overbidding (or the stronger the effect of resale).

From Proposition 3, we can obtain the expected payoff for a bidder, denoted as  $\pi(v, m, v_L, v_H)$ , when she enters the auction with  $m - 1$  other bidders:

$$\pi(v, m, v_L, v_H) = \begin{cases} \beta(v, 1, v_L, v_H) & \text{if } m = 1 \\ \int_{v_L}^v [\beta(v, m, v_L, v_H) - \beta(x, m, v_L, v_H)] f_{in, m-1}^{(1)}(x) dx & \text{if } m \geq 2 \end{cases}$$

Similarly as in the no-resale case, we can define the expected payoff for a bidder who enters the initial auction,  $\pi(v, v_L, v_H)$ , and the expected payoff for a bidder who stays out of the initial auction,  $L(v, v_L, v_H)$ .

$$\begin{aligned} \pi(v, v_L, v_H) &= \sum_{m=1}^n C_{n-1}^{m-1} (p_E)^{m-1} (1 - p_E)^{n-m} \pi(v, m, v_L, v_H) \\ L(v, v_L, v_H) &= \sum_{m=1}^{n-1} C_{n-1}^m (p_E)^m (1 - p_E)^{n-m-1} L(v, m, v_L, v_H) \\ &= \sum_{m=2}^n C_n^{m-1} (p_E)^{m-1} (1 - p_E)^{n-m} L(v, m-1, v_L, v_H) \end{aligned}$$

It can be verified that given  $(v_L, v_H)$ ,  $\pi(v, v_L, v_H) - L(v, v_L, v_H)$  is strictly increasing in  $v$ , thus again the entry equilibria are characterized by entry thresholds.

**Proposition 4** *When resale is allowed, a symmetric entry equilibrium always exists, and is unique under certain conditions. In the case with multiple equilibria, for any two pairs of entry equilibrium thresholds  $(v_{1L}, v_{1H})$  and  $(v_{2L}, v_{2H})$  where  $v_{1L} < v_{2L}$ , it must be the case that  $v_{1H} > v_{2H}$ .*

**Proof.** See Appendix. ■

So in the case of multiple equilibria, there is a natural ordering of equilibria, which is in the increasing order of entry thresholds for type- $L$  bidders (or equivalently, in the decreasing order of entry thresholds for type- $H$  bidders).<sup>13</sup> In the proof we identify a necessary and sufficient condition

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<sup>13</sup>This, we conjecture, may have to do with motives for speculative entry and resale hunting abstentions to be

for the uniqueness of the equilibrium. The condition is highly technical so we refer readers to the appendix for the details. For some limiting cases, we have identified the following sufficient conditions (each alone is sufficient to assure that the equilibrium is unique): (1)  $c_L$  and  $c_H$  are sufficiently close; (2)  $q$  is sufficiently small or sufficiently large; (3)  $n$  is sufficiently large; and (4)  $c_L$  and  $c_H$  are neither too close nor too far apart, while  $n$  and  $q$  are relatively small. Conditions (1) and (2) basically say that if bidders incur a common entry cost (or if the entry costs are almost the same), there will be no resale; hence, the entry equilibrium is also unique (a result consistent with Proposition 1). Condition (3) says that when  $n$  is sufficiently large, some type- $H$  bidder or some type- $L$  bidder with a value higher than  $v_H$  will win the item from the initial auction, which eliminates the possibility of resale. So the symmetric entry equilibrium will also be unique.

As in the benchmark case, for  $c_L$  and  $c_H$  to induce interior entry thresholds for both types, i.e.,  $0 < v_L < v_H < 1$ , bidders with type  $(v_L, c_L)$  or type  $(v_H, c_H)$  should be indifferent between entering and staying out:

$$\begin{cases} \pi(v_L, v_L, v_H) - c_L & = 0 \\ \pi(v_H, v_L, v_H) - L(v_H, v_L, v_H) - c_H & = 0 \end{cases} \quad (4)$$

In our model, we assume that a second-price sealed-bid auction is conducted in the initial auction. This turns out to be without loss of generality, as in our setting the revenue equivalence holds among all standard auctions (in which the bidder with the highest bid wins the item, and the bidder with the lowest possible value earns zero expected payoff). The reason is that all standard auctions will induce the same entry thresholds, which guarantees the symmetric IPV setting for revenue equivalence to hold.

**Proposition 5** *Given that the reseller runs an optimal auction in the resale stage, revenue equivalence holds among all standard auctions conducted in the initial auction in the symmetric (entry and bidding) equilibrium.*

**Proof.** See Appendix. ■

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introduced later. A lower  $v_L$  implies more speculative entry, and a higher  $v_H$  implies more resale hunting abstentions. Thus, it seems intuitive that if  $v_L$  decreases,  $v_H$  should increase.

## 4 Resale versus No Resale: A Comparison

In general, it is very difficult to obtain closed-form solutions for the equilibrium in both the no-resale and resale cases; nonetheless we can establish the following comparison results.

**Proposition 6** *For any pair of  $(c_L, c_H)$  that induces at least one pair of interior equilibrium thresholds in both the no-resale and resale cases,  $\tilde{v}_L > v_L$  and  $\tilde{v}_H < v_H$ .*

**Proof.** See Appendix. ■

While resale affects bidding only for type- $L$  bidders, it affects entry thresholds for both types. Proposition 6 suggests that when resale is allowed, the entry threshold is lower for type- $L$  bidders but higher for type- $H$  bidders. In other words, when resale is allowed, it is more likely for the low-cost type to enter the auction, while it is more likely for the high-cost type to stay away from the auction. For the low-cost type, bidders with  $v \in (v_L, \tilde{v}_L)$  can be referred to as *entry speculators*, since they enter the auction simply because the resale opportunity is available. On the other hand, for the high-cost type, bidders with  $v \in (\tilde{v}_H, v_H)$  can be referred to as *resale hunters*, as they stay out only because the resale opportunity is available and they might be able to obtain the item from the post-auction resale.

Analytical solutions are in general unavailable in our model, so it would be difficult to obtain general comparative statics results. However, some limiting cases should be informative. For example, when  $q = 0$  or  $1$ , bidder heterogeneity disappears and all bidders will follow a common entry threshold, in which case resale never occurs and speculative entry and resale hunting will both be absent in equilibrium. So the effect of changing  $q$  on the extent of speculative entry or resale hunting must be non-monotonic. That is, it must rise starting from  $q = 0$  and fall toward  $q = 1$ . A similar non-monotonicity with respect to  $n$  should be anticipated as well, as when  $n = 1$  or when  $n$  is sufficiently large, resale will not occur either in equilibrium.

In order to gain some insights about how resale affects entry, efficiency, and expected revenue, we now work with a specific value distribution, i.e., the case where  $F(\cdot)$  is the uniform distribution. Obviously, any truncated distribution of the uniform distribution satisfies the monotone hazard rate property.

For simplicity we will focus on the interior entry thresholds in both benchmark and resale cases. Under the uniform distribution, since all truncated density functions are constant, there exist closed-

form expressions (polynomials) for the indifference conditions. With uniform distribution, the indifference conditions (1) and (4), respectively, are as follows:

$$\begin{cases} [q\tilde{v}_L + (1-q)\tilde{v}_H]^{n-1}\tilde{v}_L & = c_L \\ [q\tilde{v}_L + (1-q)\tilde{v}_H]^{n-1}\left[\tilde{v}_H - \frac{1}{nq}\right] + \frac{(\tilde{v}_H)^{n-1}}{nq} & = c_H \end{cases}$$

and

$$\begin{cases} \frac{(n-2)[qv_L+(1-q)v_H]^n}{n(1-q)} - \frac{qv_L[qv_L+(1-q)v_H]^{n-1}}{(1-q)} + \frac{2\left[\frac{1+q}{2}v_L+\frac{1-q}{2}v_H\right]^n}{n(1-q)} & = c_L \\ -\frac{[qv_L+(1-q)v_H]^n}{n(1-q)} + [qv_L+(1-q)v_H]^{n-1}v_H + \frac{(v_H)^n}{n(1+q)} + \frac{2q\left[\frac{1+q}{2}v_L+\frac{1-q}{2}v_H\right]^n}{n(1-q^2)} & = c_H \end{cases}$$

For expected revenue, we have

$$ER^{NR}(n) = \sum_{m=1}^n C_n^m \tilde{p}_E^m (1 - \tilde{p}_E)^{n-m} \int_{\tilde{v}_L}^1 x \tilde{f}_{m,in}^{(2)}(x) dx \quad (5)$$

$$ER^R(n) = \sum_{m=1}^n C_n^m p_E^m (1 - p_E)^{n-m} \int_{v_L}^1 \beta(x, m) f_{m,in}^{(2)}(x) dx \quad (6)$$

where  $\beta(x, m)$  is the bidding function when resale is allowed in the initial auction conditional on entry, where the first argument is the value of the bidder and the second argument is the number of entrants:  $\beta(x, m) = x$  if  $x > v_H$ , and  $\beta(x, m) = w(x, m, v_L, v_H)$  if  $x \leq v_H$ .

Note that the expressions (5) and (6) suggest that the expected revenues depend on the entry probability ( $\tilde{p}_E$  and  $p_E$ ) and the revenues conditional on  $m$  entrants. Clearly, if  $p_E > \tilde{p}_E$ , then the (binomial) distribution of  $m$  induced by  $p_E$  first-order stochastically dominates that induced by  $\tilde{p}_E$ ; that is, there will be more entrants on average in the resale case. On the other hand, although we have shown that  $\tilde{v}_L > v_L$  and  $\beta(x, m) \geq x$ ,  $f_{m,in}^{(2)}(x) < \tilde{f}_{m,in}^{(2)}(x)$  when  $x$  is relatively large, as some high-value type- $H$  bidders stay out when resale is allowed. Therefore, the effect of resale on expected revenue is not clear.

We define efficiency to be the expected total surplus generated from the sale (i.e., the expected value of the final item owner less the expected total entry costs incurred):

$$\begin{aligned}
ES^{NR}(n, q, c_L, c_H) &= \sum_{m=1}^n C_n^m (\tilde{p}_E)^m (1 - \tilde{p}_E)^{n-m} \int_{\tilde{v}_L}^1 x \tilde{f}_{m,in}^{(1)}(x) dx - n \tilde{p}_E \bar{c}^{NR}, \text{ and} \\
ES^R(n, q, c_L, c_H) &= \sum_{m=1}^n C_n^m p_E^m (1 - p_E)^{n-m} \cdot \\
&\quad \left\{ \int_{v_H}^1 x f_{m,in}^{(1)}(x) dx + \int_{v_L}^{v_H} \left[ \int_{r(x)}^{v_H} z f_{out,n-m}^{(1)}(z) dz + \int_{v_L}^{r(x)} x f_{out,n-m}^{(1)}(z) dz \right] f_{m,in}^{(1)}(x) dx \right\} \\
&\quad - n p_E \bar{c}^R,
\end{aligned}$$

where  $\bar{c}^{NR}$  is the expected entry cost of an entrant when resale is not allowed and  $\bar{c}^R$  is the expected entry cost of an entrant when resale is allowed (so, for example,  $\bar{c}^{NR} = [q(1 - F(\tilde{v}_L))c_L + (1 - q)(1 - F(\tilde{v}_H))c_H]/\tilde{p}_E$ ). The terms in  $ES^R$  above reflect the several events for the final allocation: (1) there is no resale, (2) there is resale but no outside bidder can submit a bid higher than or equal to the optimal reserve price, and (3) at least one outside bidder can submit a bid higher than or equal to the optimal reserve price.

Clearly, if the bidder with the highest value happens to be of type  $L$ , allowing resale cannot further improve the efficiency; in all the other cases, allowing resale can only reduce the efficiency when the bidder with the highest value stays out of the initial auction and, in addition, does not obtain the item in resale (because either there is no resale or the reserve price is too high). Moreover, the comparison between the expected total entry costs is also not clear. These suggest that the effect of resale on efficiency may not be clear either.

Next we present some results from our numerical analysis. The set of parameters contains  $n, q, c_L,$  and  $c_H$ . We pick  $n$  to be any integer between 2 and 10,  $q$  to be 0.1, 0.2, ..., 0.8, 0.9.  $c_L$  and  $c_H$  take values from 0 to 1 ( $c_L < c_H$ ), with the step of 0.1. The numerical results reveal the following pattern:

For any pair of  $(c_L, c_H)$  that can induce at least one pair of interior entry thresholds in both the no-resale and resale cases, we have:

1. There exists a unique pair of  $(v_L, v_H)$  as the equilibrium threshold in the resale cases;
2.  $\tilde{v}_L > v_L$  and  $\tilde{v}_H < v_H$ , which is a nice illustration of Proposition 5.
3. The expected entry probability and efficiency are always higher in the resale case, compared to the no-resale benchmark. Xu et al. (2010) also find that expected efficiency is always higher

when resale is allowed. However, in their setting this result may be due to the assumption of complete information in the resale stage, where resale always occurs whenever there is a positive gain from resale. Since this result stays the same as we extend the complete information to incomplete information setting for the resale stage, we can conclude that the resale opportunity, rather than its information structure, is more important for efficiency enhancement.

4. The auctioneer's expected revenue is lower when resale is allowed for almost all the cases (a few exceptions occur when  $c_L = 0.1, c_H = 0.2$ ,  $n$  is from 5 to 7,  $q$  is of some median value). Since  $p_E > \tilde{p}_E$  in all the cases, the only force that may hurt the revenue when resale is allowed is that some high-value and high-cost bidders (resale hunters) are replaced by low-value and low-cost bidders (entry speculators). When this force becomes strong enough, the expected revenue is reduced when resale is allowed. For most of the parameter values in this example, our result suggests that although the resale opportunity enhances competitive bidding, it fails to compensate the revenue loss due to the abstentions from those bidders with high values (and high entry cost).

We also have the following comparative static results with respect to parameters  $q$  and  $n$ :

1. The auctioneer's expected revenue increases in  $q$ . Note that when  $q = 0$ , all potential bidders are of type  $H$  and the unique entry threshold is higher than that when  $q = 1$  where all potential bidders are of type  $L$ . When  $q$  increases, there are two opposing effects on revenue: the positive effect is that the portion of type- $L$  bidders increases, while the negative effect is that as the portion of type- $H$  bidders decreases, there is less gain from resale thus entrants bid less aggressively (which also pushes up  $v_L$ ). It turns out that the positive effect dominates and the net effect is to raise the expected revenue.
2. The effect of changing  $n$  on the auctioneer's expected revenue is ambiguous. When there is no resale, an increase in the number of bidders clearly has a positive effect on the revenue. However, when there is resale, we have a few exceptions (e.g.,  $n$  is large,  $q$  is 0.1 or 0.2, and  $c_L$  and  $c_H$  are large and very close to each other, and the induced  $v_L$  and  $v_H$  are sufficiently high), where the auctioneer's expected revenue decreases in  $n$ . Again there are two opposing effects. The negative effect is as follows: when  $v_L$  and  $v_H$  are sufficiently high and  $q$  is very

small, it is very unlikely that a bidder of type  $H$  will enter the initial auction (most of  $v_H$  in those exceptions are higher than 0.96); moreover, in those exceptions,  $v_H$  increases in  $n$  while  $v_L$  decreases in  $n$ , which further reduces the chance for a type- $H$  bidder to enter. The positive effect is that the chance for a bidder of type  $L$  to enter is much higher (when  $q$  is small and  $v_H$  is high, the speculative motive is very strong and  $v_L$  decreases in  $n$ ). However, when  $q$  is really small while  $v_H$  is very high, the negative effect dominates, and the net effect is to reduce the expected revenue.

## 5 Discussion

Our analysis relies on several key assumptions. First, we assume that there is no participation cost for resale. The omission of entry costs for resale would be easier to justify in the contexts where the participation cost for resale is on a much smaller scale compared to the entry cost for the initial auction. This would be the case, for example, when the initial auction is conducted by some public authority that usually imposes stringent qualification requirements for bidding, while the resale is organized by some private parties, which is presumably more decentralized and less costly to participate.

This being said, we assume away the entry cost for resale,  $c_R$ , mainly for tractability; otherwise our analysis can easily get very involved. With  $c_R > 0$ , all the endogenously determined variables, including the equilibrium entry thresholds to the initial auction  $v_L$  and  $v_H$ , will also depend on  $c_R$ . The other complication is that the resale mechanism is now an auction with costly entry as well. Since optimal auctions with costly entry are typically different from optimal auctions *a la* Myerson, the optimal reserve price set by the reseller should not only take into account the traditional tradeoff between efficiency and rent extraction, but should also induce the optimal entry, which is characterized by the entry threshold for the resale,  $v_R$ . To see how complicated the analysis of the resale subgame would be, note that  $v_R$  depends on, in particular,  $v_w$ , the reseller's value, and  $m$ , the number of the entrants in the initial auction.<sup>14</sup>

Also note that with costly entry to resale, a necessary condition for resale to occur is that

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<sup>14</sup>When resale is initiated, the reseller announces a reserve price (based on  $v_w$  and  $m$ ). The potential buyers will then make their entry decisions simultaneously and independently.

$v_w < v_H - c_R$ ; otherwise, resale will never be initiated. Given this, the equilibrium bid function in the initial auction (3) should be revised as follows:

$$\beta(v, m, v_L, v_H) = \begin{cases} w(v, m, v_L, v_H) & \text{if } v < v_H - c_R \\ v & \text{if } v \geq v_H - c_R \end{cases}$$

That is, a bidder with a value higher than  $v_H - c_R$  should not anticipate any benefit from resale and hence bids her value in the initial auction.

A concern, with  $c_R > 0$ , regards whether resale will be completely shut down (i.e.,  $v_R = v_H$ ). We will demonstrate that this concern, while legitimate, is unwarranted as long as  $c_R$  is given such that  $v_H - c_R > v_R$ .<sup>15</sup> To show this claim, suppose in negation  $v_R = v_H$ , i.e., only type  $v_H$  bidder would enter the resale. Given  $v_w \in [v_L, v_H - c_R)$ , the reseller will set an optimal reserve price  $r^*(v_w) \in [v_w, v_H - c_R)$  (otherwise no entry can possibly occur).<sup>16</sup> But this means that an outsider with value  $v_H$  can enter to win the item with a strictly positive profit, contradicting the assumption that  $v_R = v_H$ . This suggests that in equilibrium,  $v_R < v_H$ , which in turn implies that entry to resale must occur with positive probability.

So, in equilibrium, resale will be initiated whenever a type- $L$  bidder with a value  $v_w \in [v_L, v_H - c_R)$  wins the item, and the potential buyers would be type- $H$  bidders with values in  $(v_R, v_H]$ .

Other than the technical complications, with costly entry to resale, the entry equilibrium should still be characterized by entry thresholds, and the motives for speculative entry and resale hunting should still be present. For this reason we believe that the main insights of our analysis should carry over, even if we can manage to work out a model with costly entry to resale.

Our analysis has also been focusing on symmetric entry equilibria, in which bidders of the same entry cost type follow the same entry threshold. As first demonstrated by Samuelson (1985) and Stegeman (1996), asymmetric entry equilibria, in which bidders of the same entry cost type follow different entry thresholds, may also exist. Following Stegeman (1996), conditions for the existence of asymmetric entry equilibria are identified by Campbell (1998) and Tan and Yilankaya (2006) under a second-price auction, and by Cao and Tian (2010) under a first-price auction. While the previous work focuses on the case in which all the bidders have the same (and publicly known) entry cost,

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<sup>15</sup>When  $c_R = 0$ ,  $v_R = 0$ , and this condition holds trivially. By continuity, there exists an entry cost cutoff  $c_R^* > 0$ , such that  $v_H - c_R > v_R$  holds whenever  $0 < c_R \leq c_R^*$ .

<sup>16</sup>The dependence on  $m$  is omitted for brevity here.

we focus on the case in which entry costs are private information and may differ among different bidders. However, the essence of their analysis should carry over, and we believe that asymmetric entry equilibria should also exist in our setting.

In the presence of multiple equilibria, coordination on entry is desirable. In fact, Stegeman (1996) shows that the second-price auction achieves the highest efficiency level as long as bidders can coordinate on an asymmetric entry equilibrium. Campbell (1998) further demonstrates that, if preplay communication is possible, bidders' expected payoffs can be improved in some cheap-talk equilibrium. The idea is simple. Whenever a bidder's value realization is lower than her entry threshold, she would have incentive to reveal that information (and vice versa) in the cheap talk stage. We believe that this sort of simple and credible communication should be effective particularly for our case with multiple symmetric entry equilibria. For ease of illustration, we consider a simplest case where there are only two potential bidders,  $n = 2$ . Suppose there are two entry equilibria, characterized by  $(v_{1L}, v_{1H})$  and  $(v_{2L}, v_{2H})$ , where, without loss of generality,  $v_{1L} < v_{2L} < v_{2H} < v_{1H}$  (Proposition 4). Suppose the realized types are  $(c_L, v_1)$  and  $(c_H, v_2)$ , where  $v_{2L} < v_2 < v_{2H}$  and  $v_{1L} < v_1 < v_{2L}$ . If both bidders coordinate on the second equilibrium, no one enters and the payoffs are both zero. However, if bidder 2 signals that she is not entering anyway in a preplay cheap-talk game (which is incentive compatible as doing so never hurts and can only benefit), then bidder 1 will enter for sure (who will then win the auction and resell the item to bidder 2). Clearly, in this example cheap talk allows bidders to coordinate on entry and secure the best outcome. As illustrated by this simple example, the resale opportunities reinforce the effectiveness of the type of cheap talk as proposed in Campbell (1998).<sup>17</sup>

Finally, we assume that entry costs and valuations are statistically independent. So, we do not allow for correlations between bidders' two-dimensional types. One justification is that there does not seem to be a consensus in the literature over whether the values and entry costs should be positively or negatively correlated. But if for some reason we believe that valuations and entry costs are positively correlated then this may be captured by a model where the values of type- $H$  bidders

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<sup>17</sup>We are not aware of any existing work on communicating entry costs via cheap talk. A closely related idea is to run a pre-auction mechanism to elicit truthful reports on entry costs, and the set of final bidders are then determined contingent on the reported entry cost types. Lu and Ye (2010) provide such an analysis in a model where bidders incur private entry costs before they learn their valuations.

are drawn from  $F(\cdot|c = c_H)$  and the values of type- $L$  bidders are drawn from  $F(\cdot|c = c_L)$ , where  $F(\cdot|c = c_H)$  first-order stochastically dominates  $F(\cdot|c = c_L)$ . Such an environment should make a stronger case for our main results as it will strengthen both the speculative and resale hunting incentives.

In Xu et al. (2010), we analyze a case where bidders' types  $(c, v)$  follow some general joint distribution, allowing for any correlation between  $c$  and  $v$ . However, the analysis is only tractable when we restrict our analysis to the case with  $n = 2$  and assume the complete information setting for the resale stage. Since all the main results in Xu et al. (2010) carry over to this current research, we believe that our main results should be robust to settings allowing for correlation between valuations and entry costs.

## 6 Concluding Remarks

We explicitly take into account the possibility of post-auction resale in an auction model with entry where bidders possess two-dimensional private information. We demonstrate that the symmetric entry equilibrium is characterized by entry thresholds, and we identify conditions under which such an equilibrium is unique. We show that compared to the no-resale case, resale opportunities make entry more likely for bidders with low entry costs and low values, and less likely for bidders with high entry costs and high values. In other words, resale introduces motivation for both speculative entry and resale hunting abstentions.

In the example in which valuations follow a uniform distribution, our numerical results suggest that allowing resale always increases the expected entry probability and the expected efficiency, which is consistent with our general intuition about the role resale can play in auctions with costly entry. So, allowing resale is beneficial from the social point of view. However, the effects of allowing resale on expected revenue is ambiguous, although for almost all the parameter values that we chose, allowing resale hurts the initial seller.

Our analysis relies on several key assumptions. We have discussed the restrictions and argued that simplifications are made mainly for ease of analysis, without compromising main results and main insights of our analysis. Relaxation of those assumptions to allow for a more general analysis on entry and resale is left for future research.

## Appendix

**Proof of Proposition 1:** We will first consider the case with interior entry thresholds ( $0 < v_L < v_H < 1$ ) and then the case with corner solution ( $0 < v_L < v_H = 1$ ).

Define  $\tilde{Y} = \{(c_L, c_H) : \tilde{\pi}(x_1, x_1, x_2) = c_L \text{ and } \tilde{\pi}(x_2, x_1, x_2) = c_H \text{ for some } x_1, x_2 \text{ where } 0 < x_1 < x_2 < 1\}$ . That is,  $\tilde{Y}$  is the set of all pairs of  $(c_L, c_H)$  that can induce interior entry thresholds. When  $(c_L, c_H) \in \tilde{Y}$ , it is obvious that any  $(\tilde{v}_L, \tilde{v}_H)$  that solves the equation system (1) is a pair of symmetric equilibrium entry thresholds because given  $(\tilde{v}_L, \tilde{v}_H)$ , the function  $\tilde{\pi}(v, \tilde{v}_L, \tilde{v}_H)$  is strictly increasing in  $v$ . Next we show the uniqueness of equilibrium for this case. We can rewrite the equation system (1) as follows:

$$\begin{aligned}\tilde{\pi}(x_1, x_1, x_2) &= [qF(x_1) + (1-q)F(x_2)]^{n-1}x_1 = c_L \\ \tilde{\pi}(x_2, x_1, x_2) - \tilde{\pi}(x_1, x_1, x_2) &= \int_{x_1}^{x_2} [qF(z) + (1-q)F(x_2)]^{n-1}dz = c_H - c_L\end{aligned}$$

It can be easily verified that the signs of the first-order partial derivatives are given below:

$$\begin{aligned}\partial\tilde{\pi}(x_1, x_1, x_2)/\partial x_1 &> 0, \quad \partial\tilde{\pi}(x_1, x_1, x_2)/\partial x_2 > 0, \\ \partial[\tilde{\pi}(x_2, x_1, x_2) - \tilde{\pi}(x_1, x_1, x_2)]/\partial x_1 &< 0, \quad \partial[\tilde{\pi}(x_2, x_1, x_2) - \tilde{\pi}(x_1, x_1, x_2)]/\partial x_2 > 0.\end{aligned}$$

Let  $\hat{D} = \{(x_1, x_2) : 0 < x_1 \leq x_2 < 1\}$ . Applying the implicit function theorem, there exists a unique, continuously differentiable function,  $\phi(\cdot, c_L) : (0, 1) \rightarrow (0, 1)$ , such that

$\{(x_1, \phi(\cdot, c_L)) : x_1 \in (0, 1)\} = \{(x_1, x_2) \in \hat{D} : \tilde{\pi}(x_1, x_1, x_2) = c_L\}$ . In  $\hat{D}$ ,  $\phi(x_1, c_L)$  is strictly decreasing in  $x_1$ . Similarly, there exists a unique, continuously differentiable function,  $\psi(\cdot, c_H) : (0, 1) \rightarrow (0, 1)$ , such that

$\{(x_1, \psi(\cdot, c_H)) : x_1 \in (0, 1)\} = \{(x_1, x_2) \in \hat{D} : \tilde{\pi}(x_2, x_1, x_2) = c_H\}$  and  $\psi(x_1, c_H)$  is strictly increasing in  $x_1$ .

To solve the equation system (1), we can solve from  $\tilde{\pi}(\phi(x_1, c_L), x_1, \phi(x_1, c_L)) = c_H$  for  $x_1$  first, and then obtain  $x_2$  from  $x_2 = \phi(x_1, c_L)$ . Based on the signs of all the first-order partial derivatives, we have

$$\frac{d\tilde{\pi}(\phi(x_1, c_L), x_1, \phi(x_1, c_L))}{dx_1} = \frac{\partial\tilde{\pi}(x_2, x_1, x_2)}{\partial x_1}\Big|_{x_2=\phi(x_1, c_L)} + \frac{\partial\tilde{\pi}(x_2, x_1, x_2)}{\partial x_2}\Big|_{x_2=\phi(x_1, c_L)} \cdot \frac{\partial\phi(x_1, c_L)}{\partial x_1} > 0.$$

Therefore, as long as there exists at least one solution  $x_1^*$  to the equation  $\tilde{\pi}(\phi(x_1, c_L), x_1, \phi(x_1, c_L)) = c_H$ , the solution is unique. A unique  $x_1^*$  also induces a unique  $x_2^*$  because  $\partial\phi(x_1, c_L)/\partial x_1 < 0$ .

Now we consider the case  $(c_L, c_H) \notin \tilde{Y}$ , where entry thresholds involve some corner solution. In this case, the following conditions must hold:

$$\begin{cases} \tilde{\pi}(\tilde{v}_L, \tilde{v}_L, 1) = c_L \\ \tilde{\pi}(1, \tilde{v}_L, 1) \leq c_H \\ 0 \leq \tilde{v}_L < 1 \end{cases} .$$

That is, all type- $H$  bidders prefer staying out, and no bidder has an incentive to deviate from the entry thresholds  $(\tilde{v}_L, 1)$  (as given  $\tilde{v}_L$ ,  $\tilde{\pi}(v, \tilde{v}_L, 1)$  is strictly increasing in  $v$ ).

The existence of  $\tilde{v}_L$  in this case is obvious given that  $\tilde{\pi}(0, 0, 1) = 0 < c_L$  and  $\tilde{\pi}(1, 1, 1) = 1 > c_L$ . The uniqueness is also trivial as the solution of  $\tilde{v}_L$  is unique given that  $\partial\tilde{\pi}(x_1, x_1, x_2)/\partial x_1 > 0$ . ■

**Proof of Proposition 2:** Define  $\alpha(c_L) = \tilde{\pi}(1, \lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L), 1)$ , where  $\phi^{-1}(x_2, c_L) \triangleq \{x_1 : x_2 = \phi(x_1, c_L)\}$ .  $\phi^{-1}(x_2, c_L)$  exists (as  $\phi(x_1, c_L)$  is strictly decreasing in  $x_1$ ) and is continuously differentiable. We take the limit  $\lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L)$  given that the function  $\phi(x_1, c_L)$  is defined on  $\hat{D}$ , and that any point  $(x_1, 1)$  is in the hull of  $\hat{D}$ . By continuity of the function  $\phi^{-1}(\cdot, c_L)$ ,  $\lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L)$  exists and is equal to  $\phi^{-1}(\lim_{\xi \rightarrow 1-} \xi, c_L)$ .

Substituting  $\phi(x_1, c_L)$  for  $x_2$  in  $\tilde{\pi}(x_2, x_1, x_2) = c_H$ , we have  $\tilde{\pi}(\phi(x_1, c_L), x_1, \phi(x_1, c_L)) = c_H$ . When  $x_1 = \lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L)$ ,  $\tilde{\pi}(\phi(x_1, c_L), x_1, \phi(x_1, c_L)) = \tilde{\pi}(1, \lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L), 1) = \alpha(c_L) > c_H$ ; when  $x_1 = \tilde{\phi}(c_L)$ ,  $\tilde{\pi}(\phi(x_1, c_L), x_1, \phi(x_1, c_L)) = \tilde{\pi}(\tilde{\pi}(c_L), \tilde{\pi}(c_L), \tilde{\pi}(c_L)) = c_L < c_H$ . By continuity, there exists at least one solution  $x_1^* \in (\lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L), \tilde{\pi}(c_L))$ .

The next step is to plug  $x_1 = x_1^*$  into  $\phi(x_1, c_L)$  to obtain  $x_2^*$ . Note that the implicit function theorem already restricts  $(x_1, \phi(x_1, c_L))$  to the upper left triangle of the unit square. Moreover, since we have shown that by fixing  $c_L$ ,  $\phi(x_1, c_L)$  is strictly decreasing in  $x_1$  and  $x_1^* \in (\lim_{\xi \rightarrow 1-} \phi^{-1}(\xi, c_L), \tilde{\phi}(c_L))$ , we have  $x_2^* > x_1^*$ . Therefore, for any  $(c_L, c_H)$  such that  $c_L \in (0, 1)$  and  $c_H \in (c_L, \alpha(c_L))$ , there exists a unique solution in  $\hat{D}$  to the equation system (1).

No pair of  $(c_L, c_H)$  outside the set  $\{(c_L, c_H) : c_L \in (0, 1), c_H \in (c_L, \alpha(c_L))\}$  can induce interior solutions to the equation system (1). The reason is straightforward: given some  $c_L \in (0, 1)$ , it is obvious that any  $c_H$  that is strictly higher than  $\alpha(c_L)$  cannot induce a valid solution because  $\alpha(c_L)$  is the highest possible  $c_H$  (which induces  $\tilde{v}_H = 1$ ). Therefore, the set of  $(c_L, c_H)$  that can induce a pair of equilibrium thresholds in  $\hat{D}$  is equal to  $\tilde{Y}$ . ■

**Proof of Proposition 3:** Conditional on entry, no bidder has an incentive to bid more than  $\beta(v, m, v_L, v_H)$ , where  $v$  is her private value. The reason is as follows: over-bidding affects the outcome only when she wins against some other bidder whose bid is higher than  $\beta(v, m, v_L, v_H)$ , in which case this (overbidding) bidder pays an amount greater than her expected contingent payoff,  $\beta(v, m, v_L, v_H)$ , incurring a net loss.

Bidding less than  $\beta(v, m, v_L, v_H)$  is not a profitable deviation either. First of all, bidding less than  $\beta(v_L, m, v_L, v_H)$  is not profitable, because when all other entrants follow the pair of entry thresholds and use  $\beta(\cdot, m, v_L, v_H)$  as the bidding function, this amount is the lowest possible rival bid. Assume this bidder mimics type  $\hat{v}$ ,  $v_L \leq \hat{v} < v$  and that  $z$  is the value of the highest rival. Under-bidding affects the outcome only when  $\hat{v} < z < v$ , namely, when she loses while she is not supposed to.

1. When  $z \geq v_H$ , there is no resale. The bidder is worse off by deviating.
2. When  $v_H \geq z \geq r^{-1}(\min\{v, v_H\})$ , the deviant's contingent payoff from resale (as a buyer) will be 0 as  $v \leq r(z)$  almost surely,<sup>18</sup> while her equilibrium payoff will be positive.
3. When  $z < r^{-1}(\min\{v, v_H\})$ ,  $w(z, m, v_L, v_H)$  is the highest rival's bid. We will argue that given any  $z$ , there is no benefit from deviation. Note that if the bidder in question does not deviate, the expected payoff conditional on entry should be  $\beta(v, m, v_L, v_H) - w(z, m, v_L, v_H) \geq v - w(z, m, v_L, v_H)$ . If she deviates, the expected payoff is:

$$\begin{aligned}
& [F_{out}(r(z))]^{n-m}(v - r(z)) + C_{n-m}^1 [F_{out}(v) - F_{out}(r(z))] [F_{out}(r(z))]^{n-m-1} [v - E(x|r(z) \leq x \leq v)] \\
& + \int_{r(z)}^{\min\{v, v_H\}} (v - x) f_{out, n-m}^{(1)}(x) dx \\
< & [F_{out}(r(z))]^{n-m}(v - z) + C_{n-m}^1 [1 - F_{out}(r(z))] [F_{out}(r(z))]^{n-m-1} (v - r(z)) \\
& + \int_{r(z)}^{v_H} (v - x) f_{out, n-m}^{(2)}(x) dx \\
= & v - w(z, m, v_L, v_H).
\end{aligned}$$

So there is no profitable deviation for underbidding either. This shows that bidding according to  $\beta(\cdot, m, v_L, v_H)$  in the initial auction is one part of the symmetric equilibrium (of the whole game).

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<sup>18</sup>If  $v > v_H$ , we have  $v_H \geq z \geq r^{-1}(v_H) = v_H$ , which holds with probability zero.

Obviously  $w(v_H, m, v_L, v_H) = v_H$  (when the initial auction winner's value is  $v_H$ , there is no scope for resale). Applying the envelope theorem to (2), we have

$$\frac{\partial w}{\partial v} = F_{out}(r(v))^{n-m} < 1.$$

Thus  $w(v_H, m, v_L, v_H) > v$  for  $v \in [v_L, v_H)$  and  $w(v_H, m, v_L, v_H) - v$  is decreasing in  $v$ . ■

**Proof of Proposition 4:** Define  $Y = \{(c_L, c_H) : \pi(x_1, x_1, x_2) = c_L \text{ and } \pi(x_2, x_1, x_2) - L(x_2, x_1, x_2) = c_H \text{ for some } x_1, x_2 \text{ where } 0 < x_1 < x_2 < 1\}$ . That is,  $Y$  is the set of all pairs of  $(c_L, c_H)$  that can induce interior entry thresholds.

We first show that when  $(c_L, c_H) \in Y$ , any solution to the equation system (4) characterizes a pair of equilibrium thresholds for entry.

When a bidder's entry cost is  $c_L$  and her value is lower than  $v_L$ , she does not benefit from entry because given that everyone else follows the entry thresholds  $(v_L, v_H)$ , her expected payoff from entry,  $(1 - p_E)^{n-1}w(v, 1, v_L, v_H)$ , is strictly increasing in  $v$ . When a bidder's entry cost is  $c_L$  and her value is higher than  $v_L$ , her expected payoff from staying out,  $L(v, v_L, v_H)$ , is lower than her equilibrium payoff,  $\pi(v, v_L, v_H) - c_L$  (as  $\pi(v, v_L, v_H) - L(v, v_L, v_H)$  is strictly increasing in  $v$  and  $\pi(v_L, v_L, v_H) - L(v_L, v_L, v_H) = c_L$ ). Similarly, bidders with entry cost  $c_H$  do not have incentive to deviate from the entry threshold  $v_H$ .

When  $(c_L, c_H) \in (0, 1) \times (0, 1) \setminus Y$ , i.e., when  $(c_L, c_H)$  fails to induce interior entry thresholds for both types  $H$  and  $L$ , the equilibrium is determined by one of the following three cases (involving at least one corner threshold):

1. All type- $L$  bidders prefer entry ( $v_L = 0$ ):

$$\left\{ \begin{array}{ll} \pi(0, 0, v_H) & \geq c_L \\ \pi(v_H, 0, v_H) - L(v_H, 0, v_H) & = c_H \\ 0 < v_H & \leq 1 \end{array} \right. .$$

No bidder has an incentive to deviate because  $\pi(v, 0, v_H) - L(v, 0, v_H)$  strictly increases in  $v$ .

2. All type- $H$  bidders prefer staying out ( $v_H = 1$ ):

$$\left\{ \begin{array}{l} \pi(v_L, v_L, 1) = c_L \\ \pi(1, v_L, 1) - L(1, v_L, 1) \leq c_H \\ 0 \leq v_L < 1 \end{array} \right. .$$

No bidder has an incentive to deviate because  $\pi(v, v_L, 1)$  strictly increases in  $v$ .

3.  $v_L = 0$  and  $v_H = 1$ , i.e., all type- $L$  bidders enter the initial auction and all type- $H$  bidders stay out.

Hence an entry equilibrium involving corner solution always exists.<sup>19</sup> The uniqueness of equilibrium is also obvious.

We next turn to the uniqueness of the equilibrium with interior thresholds for both types. We will identify conditions under which the equilibrium is unique.

To simplify notation, we substitute  $v_L = x_1$  and  $v_H = x_2$  into the equation system (4) to obtain

$$\left\{ \begin{array}{l} \pi(x_1, x_1, x_2) - c_L = 0 \\ \pi(x_2, x_1, x_2) - L(x_2, x_1, x_2) - c_H = 0 \end{array} \right. \quad (7)$$

As in the proof of Proposition 1, we will employ the implicit function theorem. Based on the two indifference conditions in (7), we will first solve  $x_2$  in terms of  $x_1$  to obtain  $\eta(x_1, c_L)$  and  $\varphi(x_1, c_H)$ , respectively. The solution(s) for  $x_1$  is then obtained by equating  $\eta(x_1, c_L)$  and  $\varphi(x_1, c_H)$ . We will identify conditions under which the solution for  $x_1$  (and hence the solution for the equilibrium) is unique.

In (7), it is obvious that  $\partial\pi(x_1, x_1, x_2)/\partial x_1 > 0$  and  $\partial\pi(x_1, x_1, x_2)/\partial x_2 > 0$  when  $x_1 \neq x_2$ . The reason is as follows:  $\pi(x_1, x_1, x_2)$  is the expected payoff for a bidder with value  $x_1$ , when the pair of thresholds is  $(x_1, x_2)$ . Fixing  $x_2$ , when  $x_1$  increases to  $x_1 + \varepsilon$ , we have  $\pi(x_1 + \varepsilon, x_1 + \varepsilon, x_2) > \pi(x_1, x_1 + \varepsilon, x_2) > \pi(x_1, x_1, x_2)$ . The second inequality is due to two effects: first, the chance for a bidder to enter alone is higher when the lower threshold is  $x_1 + \varepsilon$ ; second, conditional on entering alone, a bidder's expected payoff from resale is higher: even if she still uses the optimal reserve price (in the resale) when the lower threshold is  $x_1$ , her payoff from resale is higher than before (with the

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<sup>19</sup>The characterization is due to the conditions  $\partial\pi(v_L, v_L, v_H)/\partial v_L > 0$  and  $\partial[\pi(v_H, v_L, v_H) - L(v_H, v_L, v_H)]/\partial v_H > 0$  when  $v_L \neq v_H$ , which are demonstrated below.

lower threshold  $x_1$ , the potential buyers in resale are those with  $c_H$  and  $v \in (x_1, x_2)$ ; with threshold  $x_1 + \varepsilon$ , in addition to those bidders, potential buyers also include the bidders with  $c_L$  and  $v \in (x_1, x_1 + \varepsilon)$ ). In the resale, the previously set reserve price can generate an expected payoff which cannot be higher than the current maximal payoff. Therefore, we have  $\pi(x_1, x_1 + \varepsilon, x_2) > \pi(x_1, x_1, x_2)$ . Fixing  $x_1$ , when  $x_2$  increases,  $\pi(x_1, x_1, x_2)$  also increases: the probability for one to enter alone increases and the expected payoff from resale conditional on entering alone also increases. We also have

$$\begin{aligned}
\partial[\pi(x_2, x_1, x_2) - L(x_2, x_1, x_2)]/\partial x_1 &= (n-1)qf(x_1)\{[qF(x_1) + (1-q)F(x_2)]^{n-2}x_2 \\
&\quad + \int_{x_1}^{x_2} [qF(x_1) + (1-q)F(r(x))]^{n-2}[r'(x) - 1]dx\} \\
&> (n-1)qf(x_1)\{[qF(x_1) + (1-q)F(x_2)]^{n-2}x_2 \\
&\quad - [qF(x_1) + (1-q)F(x_2)]^{n-2}(x_2 - x_1)\} \\
&= (n-1)qf(x_1)[qF(x_1) + (1-q)F(x_2)]^{n-2}x_1 \\
&> 0. \\
\partial[\pi(x_2, x_1, x_2) - L(x_2, x_1, x_2)]/\partial x_2 &= (n-1)(1-q)f(x_2)[qF(x_1) + (1-q)F(x_2)]^{n-2}x_2 \\
&\quad + [qF(x_1) + (1-q)F(x_2)]^{n-1} \\
&> 0
\end{aligned}$$

By the implicit function theorem, there exists a unique, continuously differentiable function  $\eta(\cdot, c_L)$ , such that  $\{(x_1, \eta(x_1, c_L)) : x_1 \in (0, 1)\} = \{(x_1, x_2) \in \hat{D} : \pi(x_1, x_1, x_2) = c_L\}$  and a unique, continuously differentiable function  $\varphi(\cdot, c_H)$ , such that  $\{(x_1, \varphi(x_1, c_H)) : x_1 \in (0, 1)\} = \{(x_1, x_2) \in \hat{D} : \pi(x_2, x_1, x_2) - L(x_2, x_1, x_2) = c_H\}$ .

Applying the chain rule, given  $c_L$  and  $c_H$ , both  $\eta(x_1, c_L)$  and  $\varphi(x_1, c_H)$  are decreasing in  $x_1$ . Since  $x_1$  is obtained by  $\eta(x_1, c_L) = \varphi(x_1, c_H)$ , without further information, the uniqueness for  $x_1$  is not straightforward. However, in the case with multiple equilibria, for any two entry equilibrium thresholds  $(v_{1L}, v_{1H})$  and  $(v_{2L}, v_{2H})$ , if  $v_{1L} < v_{2L}$ , it must be the case that  $v_{1H} = \eta(v_{1L}, c_L) > \eta(v_{2L}, c_L) = v_{2H}$ , that is, there is a natural ranking for the multiple equilibria.

Next we derive sufficient conditions under which the pair of equilibrium entry thresholds is unique. As mentioned above, to solve the equation system (7) is equivalent to identifying the intersection(s) between  $\eta(x_1, c_L)$  and  $\varphi(x_1, c_H)$  (the existence has already been established). Since both  $\eta(x_1, c_L)$

and  $\varphi(x_1, c_H)$  are, given  $c_L$  and  $c_H$ , decreasing in  $x_1$ , an immediate sufficient condition is that  $\partial\eta(x_1, c_L)/\partial x_1 < \partial\varphi(x_1, c_L)/\partial x_1$  or  $\partial\eta(x_1, c_L)/\partial x_1 > \partial\varphi(x_1, c_L)/\partial x_1$  for any  $x_1 \in (0, 1)$ .

If we define

$$\Delta(x_1, q, n, c_L, c_H) \triangleq \frac{\partial\pi(x_1, x_1, x_2)}{\partial x_1} \Big|_{x_2=\eta(x_1, c_L)} \cdot \frac{\partial[\pi(x_2, x_1, x_2) - L(x_2, x_1, x_2)]}{\partial x_2} \Big|_{x_2=\varphi(x_1, c_H)} \\ - \frac{\partial\pi(x_1, x_1, x_2)}{\partial x_2} \Big|_{x_2=\eta(x_1, c_L)} \cdot \frac{\partial[\pi(x_2, x_1, x_2) - L(x_2, x_1, x_2)]}{\partial x_1} \Big|_{x_2=\varphi(x_1, c_H)},$$

the necessary and sufficient condition for the uniqueness is that  $\Delta(x_1, q, n, c_L, c_H)$  does not change the sign at any  $x_1$  where  $\eta(x_1, c_L) = \varphi(x_1, c_H)$  (otherwise there would be multiple solutions for  $x_1$ ).

Hence any condition under which

$$\bar{\Delta}(x_1, x_2, q, n) \triangleq \frac{\partial\pi(x_1, x_1, x_2)}{\partial x_1} \cdot \frac{\partial[\pi(x_2, x_1, x_2) - L(x_2, x_1, x_2)]}{\partial x_2} \\ - \frac{\partial\pi(x_1, x_1, x_2)}{\partial x_2} \cdot \frac{\partial[\pi(x_2, x_1, x_2) - L(x_2, x_1, x_2)]}{\partial x_1}$$

does not change sign for any  $(x_1, x_2) \in \hat{D}$  is a sufficient condition for the uniqueness. Moreover, if  $\bar{\Delta}(x_1, x_2, q, n)$  does not change sign for any  $(x_1, x_2) \in \hat{D}$ , where  $\hat{D} \subset \hat{D}$ , we have that the corresponding set of  $(c_L, c_H)$  generates a unique pair of equilibrium thresholds.

We do not give the expression for  $\bar{\Delta}(x_1, x_2, q, n)$  here because doing so would be tedious. Nevertheless we can evaluate the following cases:

1. When  $c_L = c_H$ ,  $\eta(x_1, c_L) = \varphi(x_1, c_H) = x_1$  and  $x_2 = x_1$ :

$$\bar{\Delta}(x_1, x_2, q, n) = (1 - p_E)^{n-1} \frac{\partial\pi(x_1, x_1, x_2)}{\partial x_1} \\ + (n-1)(1-q)f(x_2)(1-p_E)^{n-2}[qF(x_1) + (1-q)F(r(x_1))]^{n-1}x_2 \\ > 0.$$

This condition implies that when  $c_L$  and  $c_H$  are sufficiently close, the solution to the equation system (4) is unique.

2. We also have

$$\lim_{q \rightarrow 0} \bar{\Delta}(x_1, x_2, q, n) > (n-1)f(x_2)(1-p_E)^{n-2}[qF(x_1) + (1-q)F(r(x_1))]^{n-1}x_2 > 0 \\ \lim_{q \rightarrow 1} \bar{\Delta}(x_1, x_2, q, n) = \lim_{q \rightarrow 1} (1-p_E)^{n-1} \frac{\partial\pi(x_1, x_1, x_2)}{\partial x_1} > 0$$

These two conditions imply that when  $q$  is sufficiently small or sufficiently large,  $\Delta(x_1, q, n, c_L, c_H) > 0$ .

3. Taking the limit, we have<sup>20</sup>

$$\begin{aligned} \lim_{n \rightarrow \infty} \bar{\Delta}(x_1, x_2, q, n) &= \lim_{n \rightarrow \infty} -(n-1)^3 q(1-q) f(x_1) f(x_2) (1-p_E)^{n-2} x_2 \\ &\quad \cdot \int_{x_1}^{x_2} [qF(x_1) + (1-q)F(r(x))]^{n-2} r'(x) \left[ \frac{(1-p_E)}{qF(x_1) + (1-q)F(r(x))} - \frac{n-2}{n-1} \right] dx \\ &= 0_- \end{aligned}$$

This condition implies that when  $n$  is sufficiently large,  $\bar{\Delta}(x_1, x_2, q, n) < 0$ .

4. Define  $M(x_1, x_2, n) = \int_{r(x_1)}^{x_2} \left[ (n-1) \frac{qF(x_1) + (1-q)F(x_2)}{qF(x_1) + (1-q)F(r(x))} - (n-2) \right] \frac{f(x_1)}{f(x)} dx$ . When the following inequality holds:

$$\begin{aligned} &\int_{x_1}^{x_2} [qF(x_1) + (1-q)F(r(x))]^{n-2} [1 - r'(x)] dx \\ &\leq (1-p_E)^{n-2} x_2 \int_{x_1}^{x_2} \left[ \frac{qF(x_1) + (1-q)F(x_2)}{qF(x_1) + (1-q)F(r(x))} - \frac{n-2}{n-1} \right] dx, \end{aligned}$$

$\bar{\Delta}(x_1, x_2, q, n) > 0$  as long as  $(1-p_E)^{n-1} / [qF(x_1) + (1-q)F(r(x_1))]^{n-1} \leq (1-q) / [qM(x_1, x_2, n)] + 1$ . These two conditions can be satisfied when  $x_1$  and  $x_2$  are not too close nor too far apart (which also means that  $c_L$  and  $c_H$  are not too close nor too far apart), and  $q, n$  are relatively small. ■

**Proof of Proposition 5:** We will show that in the symmetric (entry and bidding) equilibrium, all standard auctions will lead to the same expected payoffs (revenue equivalence).

First we claim that given the same induced entry thresholds, say,  $(v_L, v_H)$ , revenue equivalence holds. To see this, given  $m$ , the number of entrants induced by the thresholds  $(v_L, v_H)$ , the reseller faces  $n-m$  outside bidders with values independently drawn from  $F_{out}(\cdot)$ . As the resale mechanism is fixed (which is an optimal auction), the reseller's expected payoff from resale is the same. Similarly, for outside bidders, the expected payoff conditional on staying out is also invariant across different (standard) auction formats. In the initial auction, the  $m$  entrants' values are independently drawn from  $F_{in}(\cdot)$ . Hence the symmetric IPV paradigm applies and the revenue equivalence follows. Since

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<sup>20</sup>Several other terms are omitted below, as those terms converge to 0 much faster.

revenue equivalence holds given any realized  $m$ , it also holds *ex ante* (before  $m$  is realized); in other words, an auction induces the same expected payoffs as long as it induces the same entry thresholds.

But auctions with the same expected payoffs (for the seller and bidders) will indeed induce the same symmetric entry thresholds. Thus the claim established in the preceding paragraph and the same entry thresholds are self-enforcing, implying that revenue equivalence holds in the symmetric (entry and bidding) equilibrium among all standard auctions. ■

**Proof of Proposition 6:** We show that for all possible equilibrium thresholds in the resale case,  $(\tilde{v}_L, \tilde{v}_H)$ , it must be the case that  $\tilde{v}_L > v_L$  and  $\tilde{v}_H < v_H$  (we have showed that  $(\tilde{v}_L, \tilde{v}_H)$  is unique).

We will proceed by ruling out all the other cases:

1.  $\tilde{v}_L \leq v_L$  and  $\tilde{v}_H \leq v_H$ . If this is true, we immediately have  $1 - \tilde{p}_E \leq 1 - p_E$ . This implies that  $c_L = (1 - \tilde{p}_E)^{n-1} \tilde{v}_L \leq (1 - p_E)^{n-1} v_L < (1 - p_E)^{n-1} w(v_L, 1, v_L, v_H) = c_L$ , a contradiction.
2.  $\tilde{v}_L \geq v_L$  and  $\tilde{v}_H \geq v_H$ . Because  $\tilde{v}_H \geq v_H$ , we know the bidder with type  $(c_H, v_H)$  has a higher payoff in the resale than in the no-resale case. However, this cannot happen when  $\tilde{v}_L \geq v_L$  and  $\tilde{v}_H \geq v_H$  both hold. To see why, if  $\tilde{v}_L \geq v_L$  and  $\tilde{v}_H \geq v_H$ , there will be more entrants with resale; Moreover, type- $H$  bidders bid their values, which is the same as in the no-resale case, while type- $L$  bidders bid more aggressively (as they bid an amount equal to the expected payoffs from resale, which is higher than their values). Therefore, the expected payoff conditional on entry for the bidder with  $v_H$  can only be lower in the resale case, a contradiction.
3.  $\tilde{v}_L \leq v_L$  and  $\tilde{v}_H \geq v_H$ . Since  $\tilde{v}_L \leq v_L$ , we have  $\pi(\tilde{v}_L, v_L, v_H) \leq c_L$ . Since  $\pi(\tilde{v}_L, v_L, v_H) = (1 - p_E)^{n-1} w(\tilde{v}_L, 1, v_L, v_H)$  and  $w(\tilde{v}_L, 1, v_L, v_H) > \tilde{v}_L$ , we have  $(1 - p_E)^{n-1} \tilde{v}_L < \pi(\tilde{v}_L, v_L, v_H) \leq c_L = (1 - \tilde{p}_E)^{n-1} \tilde{v}_L$ . We thus have  $p_E > \tilde{p}_E$ . Next we consider the equilibrium payoff for a bidder of type  $(\tilde{v}_H, c_H)$  in the resale case. Her expected payoff conditional on entry is  $\tilde{\pi}(\tilde{v}_H, \tilde{v}_L, \tilde{v}_H)$  in the no-resale case and is  $\pi(\tilde{v}_H, v_L, v_H)$  in the resale case, while  $\pi(\tilde{v}_H, v_L, v_H) \geq c_H$  and  $\tilde{\pi}(\tilde{v}_H, \tilde{v}_L, \tilde{v}_H) = c_H$ . This implies that by entering the auction, this bidder will have a higher expected payoff in the resale case. However, we will argue that bidders bid more aggressively in the resale case. First, in the resale case type- $L$  bidders with  $v \in (v_L, \tilde{v}_H)$  bid their expected payoff from resale instead of their own values, and these bidders also enter the auction in the no-resale case. Second, type- $L$  bidders with  $v \in (\tilde{v}_L, v_L)$  stay out in the

resale case. However, this effect of reduced competition is more than offset by the entry of type- $H$  bidders with  $v \in (v_H, \tilde{v}_H)$ , as in probability sense, there will be more type- $H$  bidders with  $v \in (v_H, \tilde{v}_H)$  than type- $L$  bidders with  $v \in (\tilde{v}_L, v_L)$ , given  $p_E > \tilde{p}_E$  established above. Therefore, the bidder with  $\tilde{v}_H$  faces more entry by other bidders and will make a strictly higher expected payment in every winning event, contradicting  $\pi(\tilde{v}_H, v_L, v_H) \geq \tilde{\pi}(\tilde{v}_H, \tilde{v}_L, \tilde{v}_H)$ . ■

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