Efficient and Optimal Mechanisms with Radio Spectrum Sharing*

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Abstract

Optimal mechanism design with spectrum sharing differs from the traditional setting mainly in that some technological constraints (e.g., non-interference constraints) need to be taken into account explicitly. In this paper we characterize surplus-maximizing (efficient) and revenue-maximizing (optimal) mechanisms in a spectrum sharing context where a principal allocates transmitted power among a group of potentially interested users (transmitter-receiver pairs). Under regularity conditions about value distributions and non-interference constraints, we show that efficiency (optimality) typically involves spectrum sharing by multiple users, and the exact allocation of transmitted power is determined such that the ratio of marginal value (virtual value) over marginal cost (in terms of the cost to the interference generated) is equal among all shared users. We show that efficient and optimal mechanisms in our setting are actually dominant-strategy incentive compatible, and that they can also be implemented by well-designed all-pay or discriminatory-price auctions.

Keywords: Spectrum sharing; spectrum sharing mechanisms; efficiency; optimality.

1 Introduction

With the ever-increasing demand for wireless communications, spectrum scarcity and efficient use of wireless spectrum is becoming a major challenge. The Federal Communication Commission (FCC) has reported that the conventional fixed spectrum assignment is no longer capable of meeting today's wireless

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spectrum requirements. By 2020, 20 billion devices will be online – up from 8 billion today. To handle this explosive growth, the capacity of wireless networks must triple over the next four years.[1] Without additional airwaves to handle the traffic, consumers will face more dropped calls, connection delays, and slower downloads of data. Also, according to the spectrum usage measurements by the FCC's Spectrum Policy Task Force, many of the allocated spectrum bands are idle most of the time or not used in some area.[2] A promising approach to improve spectrum utilization is spectrum sharing, in which unlicensed secondary users are allowed to utilize the radio spectrum owned by a primary owner.[3]

For this purpose, designing a spectrum sharing mechanism that can efficiently or optimally allocate the spectrum bands or power to secondary users seems imperative. A well designed market mechanism is also necessary to provide sufficient incentives for both the primary owner and secondary users to participate in spectrum sharing.

A key challenge in economic/market analysis of spectrum sharing lies in the technological constraints implied from spectrum sharing. For example, "sensing" applications such as radar (e.g. "radiolocation") and microwave radiometry (i.e. "Earth exploration satellite-passive" or "radio astronomy") have requirements for spectral access that differ from standard communications applications. As both radar and radiometer systems are vitally important for many "public good" applications (including weather monitoring, astronomy, air traffic control, defense, etc.), spectrum sharing with commercial users should only be allowed when performance in existing applications is not compromised. In fact, FCC Commissioner Michael O'Rielly openly called for more spectrum sharing studies to help mitigate interference on increasingly crowded airwaves.[4]

In this paper, we explore socially efficient and revenue-maximizing (optimal) mechanisms with spectrum sharing. The efficient and optimal mechanism design in such a setting differs from the traditional mechanism design mainly in that some technology constraints should be explicitly taken into account. In this paper, we study a simple setting where the constraint is captured by a single non-interference constraint: the *interference temperature* in the spectrum band should be kept under some threshold, where interference temperature is defined to be the RF power measured at a receiving antenna per unit bandwidth. This is the case, for example, in the sharing of spectrum bandwidth currently occupied by

[1] "Bringing the Sharing Economy to the Airwaves Will Boost Your Bandwidth," by Kurt Schaubach, *The IEEE Spectrum Newsletters*, Jan. 29, 2018.

[2] "Report of the spectrum efficiency working group," FCC Spectrum Policy Task Force, 2002.

[3] Spectrum sharing is greatly facilitated by cognitive radio networks (Akyildiz et al., 2007). Advances in machine learning algorithms and cloud computing in recent years have allowed us to create scalable software that makes real-time decisions on spectrum sharing possible.

[4] "FCC's O'Rielly Calls for More Spectrum Sharing Studies," by Diana Goovaerts, Spectrum Week, June 28, 2017.

microwave radiometry.

Our analysis is based on a system model introduced by Huang et al. (2006), which is motivated by the scenario where (secondary) users would like to purchase a data service. The spectrum to be allocated is originally licensed to an independent entity (a private firm or a government agency), either of which we refer to as a *principal*. Users may transmit to receivers at different locations or co-located receivers at a single access point. In both cases, the principal controls the amount of bandwidth and transmitted power assigned to each user in order to keep the interference temperature at a given measurement point below a certain threshold. As in Huang et al., we also assume that all users adopt a spread spectrum signaling format, in which the transmitted power is evenly spread across the entire band controlled by the principal. This allows efficient multiplexing of data streams from different sources of different applications, and reduces the combined power-bandwidth allocation problem to a transmit power allocation problem.

Given that the object to be allocated in our model (the power) is divisible, our approach is similar to a share auction (see, for example, Wang and Zender, 2002; and Back and Zender, 1993). As mentioned above, our problem differs from the traditional optimal mechanism design (*a la* Myerson, 1981) also in that some technological constraints are taken into account explicitly. Both features might potentially complicate the characterization of the efficient/optimal mechanisms. Nevertheless under some regularity conditions about the value distributions and the interference constraints we are able to fully characterize the efficient and optimal mechanisms. We show that the efficient (optimal) mechanisms have the following properties:

- The efficient (optimal) allocation rule is to maximize the value (Myersonian virtual value) adjusted by the non-interference constraint. So in particular, the user with the maximal value (virtual value) may not secure the maximum power allocation;
- The efficient (optimal) allocation in general admits multiple users sharing the same bandwidth – the power allocation/division will be adjusted so that the ratio of marginal value (virtual value) over marginal cost (in terms of the interference generated) is equal among all the users who are allocated power.

Noticeably, the efficient and optimal mechanisms we characterized are actually dominant-strategy incentive compatible, as is the case in the traditional single-unit item allocation problem *a la* Myerson (1981). This is somewhat unexpected, as the complexity of additional technological constraints does not affect the dominant strategy implementation in our setting.

We also explore implementations of efficient and optimal mechanisms. We demonstrate that a well-designed all-pay auction or discriminatory-price auction can implement the efficient and optimal power allocation rules. Basically the all-pay or discriminatory-price auction should be augmented with a pre-announced *recovery* function. After bids are collected from all users, the underlying "types" will be recovered from the bids using the recovery function. Based on the recovered type profile, the efficient or optimal allocation of power can then be implemented according to the corresponding allocation rule.

Despite the growing interests in trying to understand economic/market aspects of spectrum sharing (see, for example, Huang et al., 2006; Gandhi et al., 2008; Kash et al., 2013; Khaledi and Abouzeid, 2014; and references therein), the characterization of efficient/optimal mechanisms with spectrum sharing remains unexplored.[5] This paper represents a first attempt in answering such a question. Our paper thus contributes to the theoretical literature on mechanism design. We hope that the basic insights obtained from this research can be extended to some more general settings.

More broadly, our analysis contributes to the literature on spectrum auctions, a theoretically challenging and practically important research area in the last 20 years (see, for example, Ausubel and Milgrom, 2002; Cramton, Shoham, and Steinberg, 2006; and Cramton, 2013). Our paper differs from most works in this literature in that we focus on transmit power allocation problem, rather than the bandwidth/frequency allocation problem.

The rest of the paper is organized as follows. Section 2 lays out a system model. Section 3 characterizes the surplus-maximizing (efficient) and revenue-maximizing (optimal) mechanisms. Section 4 is a discussion on how to extend our analysis to more general settings, and Section 5 concludes.

2 The Model

We consider the following system model, which follows the one introduced by Huang et al. (2006) closely. Spectrum with bandwidth B is to be shared among N spread spectrum users, where a user refers to a transmitter and an intended receiver pair.[6] Users in our model can be, for example, communication network providers who have only secondary access to a shared portion of the spectrum. Let h_{ij} be the channel gain from user i's transmitter to user j's receiver, n_0 be the parameter that determines the

[5] In particular, Huang et al. examine two specific auction mechanisms in allocating received power. Our paper differs from theirs in that we follow the optimal mechanism design approach.

[6] The spread spectrum communications are widely used today for military, industrial, avionics, scientific, and civil uses. The applications include jam-resistant communication systems, CDMA radios, high resolution ranging (e.g., Global Positioning System (GPS)), WLAN (Wireless Local Area Networks), etc. (http://www.tutorialsweb.com/spread-spectrum/advantages-and-applications-of-ss-communications.htm).

background noise power (assumed to be the same for all users), and p_i be the assigned transmit power to user i. Then for each i, the received Signal-to-Interference plus Noise Ratio (SINR) is given by

$$\gamma_i = \frac{p_i h_{ii}}{n_0 B + \sum_{j \neq i} p_j h_{ji}}, \quad i = 1, 2, \ldots, N. \tag{1}$$

To satisfy the interference temperature constraint, the total received power at a specified measurement point must satisfy the following constraint:

$$\sum_{i=1}^{N} p_i h_{i0} \leq P, \tag{2}$$

where h_{i0} is the channel gain from user i's transmitter to the measurement point, and P is the received power threshold to prohibit interference to a primary user. We assume that h_{ij}, h_{i0}, n_0, B, and P are all primitives of the model which are common knowledge. A system model with N transmitter-receiver pairs is illustrated in figure 1.

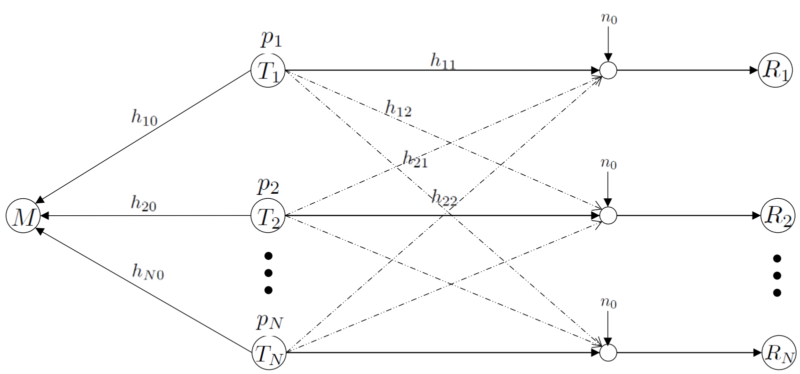

Figure 1: A System Model for N Transmitter-Receiver Pairs

Let θ_i be the user-dependent parameter that can be regarded as user i's "type," $i = 1, 2, \ldots, N$. θ_i is only known to user i, hence is user i's private information. Ex ante, we assume that θ_i's are independently and identically distributed according to a distribution function $F(\cdot)$ over support $[\underline{\theta}, \overline{\theta}]$.[7] User i's valuation of the spectrum is given by $\theta_i \log(1+\gamma_i)$, where $\log(1+\gamma_i) \equiv q_i$ is a measure for the desired Quality of Service (QoS). Using q_i, user i's valuation of the spectrum is given by $\theta_i q_i$. Thus θ_i captures user i's "marginal

[7] Our analysis allows for the more general case where θ_i is distributed according to potentially different $F_i(\cdot)$. Allowing for the heterogeneity along the value dimension would not qualitatively alter our analysis.

value" of the quality of services derived from spectrum sharing.

3 The Analysis

Under certain conditions, $\{p_i\}_{i=1}^N$ can be uniquely determined by $\{\gamma_i\}_{i=1}^N$ and hence $\{q_i\}_{i=1}^N$. In what follows we will focus on the QoS allocation problem. Once q_i's are determined, the allocated power p_i's can be recovered accordingly. First, the interference temperature constraint (2) can be reformulated in terms of q_i's:

$$H(q_1,...,q_N) \le P. \tag{3}$$

We can then define the QoS provision possibility set:

$$\mathbf{Q} = \{(q_1,...,q_N) : H(q_1,...,q_N) \le P\}.$$

For ease of analysis, we make the following assumptions:

Assumption 1. $\theta - \frac{1-F(\theta)}{f(\theta)}$ increases in $\theta \in [\underline{\theta},\overline{\theta}]$.

Assumption 2. $H(q_1,...,q_N)$ is strictly quasi-convex in $(q_1,...,q_N) \in \mathbb{R}_+^N$.

Assumption 1 is a regularity condition that ensures that bunching does not occur in the analysis of standard screening model (e.g., Myerson, 1981). Assumption 2 implies that the QoS provision possibility set is strictly convex given P. Assumption 2 is needed to ensure that the first order conditions in our analysis below are both necessary and sufficient for optimization.

By the revelation principle (Myerson, 1986), we can focus on the direct mechanism in which all users are required to report their "types." A mechanism is characterized by QoS (or equivalently, power) allocation rule $q : \Theta \to \mathbb{R}_+^N$ and payment rule $m : \Theta \to \mathbb{R}^N$, so that given a reported type profile $\widehat{\theta} = \left(\widehat{\theta}_1,...,\widehat{\theta}_N\right) \in \Theta$, the power allocated to user i is given by $q_i(\widehat{\theta})$ at a price $m_i(\widehat{\theta})$, $i = 1,2,...,N$. The direct mechanism (q,m) is incentive compatible if

$$\theta_i \in \arg\max_{\widehat{\theta}_i} E_{\theta_{-i}}\left[\theta_i q_i(\widehat{\theta}_i,\theta_{-i})) - m_i(\widehat{\theta}_i,\theta_{-i})\right], \text{ for } i = 1,2,...,N.$$

Define $Q_i(\theta_i) = E_{\theta_{-i}} q_i(\theta_i,\theta_{-i})$ and $M_i(\theta_i) = E_{\theta_{-i}} m_i(\theta_i,\theta_{-i})$ to be the interim expected allocation and payment, respectively, for user i with report θ_i (while all the other users report their types truthfully). Let $U_i(\theta_i) = \theta_i Q_i(\theta_i) - M_i(\theta_i)$ be the equilibrium (interim) expected utility for user i (with type θ_i). The following lemma is standard following the constraint simplification theorem (e.g., Milgrom, 2004):

Lemma 1. Incentive compatibility holds if and only if the following two conditions are satisfied:

1. The interim expected allocation rule $Q_i(\theta_i)$ is nondecreasing: $Q_i'(\theta_i) \geq 0$;

2. The equilibrium (interim) expected utility satisfies the envelope formula:

$$U_i'(\theta_i) = Q_i(\theta_i) \text{ or } U_i(\theta_i) = U_i(\theta^*) + \int_{\theta^*}^{\theta_i} Q_i(s)dF(s), \text{ where } \theta^* \in [\underline{\theta}, \theta_i). \tag{4}$$

To maximize social surplus (efficiency), the principal chooses the mechanism to maximize the total expected utility $\sum_{i=1}^N E_\theta[\theta_i q_i(\theta)]$ subject to individual rationality (IR), incentive compatibility (IC), and the non-interference constraint (3). To maximize expected revenue (optimality), the principal chooses the mechanism to maximize the total expected payment from the users $\sum_{i=1}^N E_\theta m_i(\theta)$ subject to the same set of constraints. We start with efficient mechanisms.

3.1 Efficient Mechanisms

The seller's objective is to find a mechanism that maximizes

$$\sum_{i=1}^N \int_{\underline{\theta}}^{\overline{\theta}} \int_{\theta_{-i}} \theta_i q_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i} f(\theta_i) d\theta_i$$

subject to the following constraints: IC: $Q_i'(\theta_i) \geq 0$ and (4), IR: $U(\theta_i) \geq 0$, for all $\theta_i \in [\underline{\theta}, \overline{\theta}]$, and ITC: interference temperature constraint: (3).

Temporarily ignoring the IR and IC constraints, we have the Lagrangian

$$L = \sum_{i=1}^N \theta_i q_i(\theta_i, \theta_{-i}) + \lambda(\theta)(P - H(q_1(\theta), ..., q_N(\theta))).$$

where $\lambda(\theta)$ is the Lagrangian multiplier for the ITC constraint, hence can be interpreted as the shadow price of the ITC constraint given θ.

Differentiating L with respect to $q_i(\theta)$, we obtain the first-order optimality condition

$$\theta_i = \lambda(\theta) H_{q_i}(q_1, ..., q_N), \; i = 1, 2, ..., N. \tag{5}$$

The interpretation of (5) is clear: at the optimum allocation $q=(q_i,q_{-i})$ the marginal social benefit (θ_i) equals the marginal social cost $(\lambda H_{q_i}(q))$ of providing q_i to user i.

Combining these N equations with the binding ITC constraint:

$$H(q_1,...,q_N)=P, \tag{6}$$

we can derive the allocation rule for the efficient auction $q^{**}(\theta)=\left(q_1^{**}(\theta),...q_N^{**}(\theta)\right)$.

Assumption 2 guarantees the interior optimum solution: $q_i^{**}(\theta)>0$ for all $i=1,2,...,N$. This also suggests that efficient allocation rule does not involve market exclusion – all users will be allocated positive amount of power. Conditions (5) also imply

$$\frac{\theta_i}{H_{q_i}\left(q_1^{**},...,q_N^{**}\right)}=\frac{\theta_j}{H_{q_j}\left(q_1^{**},...,q_N^{**}\right)} \text{ for } i,j=1,...,N,\ i\neq j. \tag{7}$$

So the efficient allocation rule requires that power be allocated in a way that the ratio of marginal surplus over marginal cost (in terms of the interference generated) should be equal across all users. While in the traditional setting without interference constraints efficient allocation should favor the ones with highest values (types), here the criterion should be based on the relative "types" that are adjusted by the interference constraints.

Note that the full market coverage (that all users are assigned positive amount of power) is due to the strict quasi-convexity of H. Should Assumption 2 fail, it is possible that efficient allocation involves the "corner" solution where some users end up with no assignment (and only those with largest marginal surplus/marginal cost ratio would be assigned positive amount of power).

Next, we identify payment rule m that truthfully implements the socially optimum allocation rule q^{**} (while satisfying IR). First, let $U_i(\underline{\theta})=0$ for all $i=1,...,N$. Then by (4), IR is satisfied for all types. Using (4) again, we have

$$E_{\theta_{-i}}[\theta_i q_i(\theta_i,\theta_{-i}))-m_i(\theta_i,\theta_{-i})]=\int_{\underline{\theta}}^{\theta_i}Q_i(s)dF(s)=E_{\theta_{-i}}\int_{\underline{\theta}}^{\theta_i}q_i(t_i,\theta_{-i})dt_i.$$

We can thus choose the following payment rule

$$m_i^{**}(\theta)=\theta_i q_i^{**}(\theta_i,\theta_{-i})-\int_{\underline{\theta}}^{\theta_i}q_i^{**}(t_i,\theta_{-i})dt_i \tag{8}$$

which by construction, satisfies (4).

Using (5) and the binding ITC, we can verify that

$$\frac{\partial q_i^{**}}{\partial \theta_i} = \frac{\Delta_{H_{-i}}}{\lambda \cdot \Delta_H}.$$

where Δ_H denotes the determinant of the Bordered Hessian matrix of H:

$$\Delta_H = \begin{vmatrix} 0 & H_1 & \cdots & H_i & \cdots & H_N \\ H_1 & H_{11} & \cdots & H_{1i} & \cdots & H_{1N} \\ \vdots & \vdots & & \vdots & & \vdots \\ H_i & H_{i1} & \cdots & H_{ii} & \cdots & H_{iN} \\ \vdots & \vdots & & \vdots & & \vdots \\ H_N & H_{N1} & \cdots & H_{Ni} & \cdots & H_{NN} \end{vmatrix} \tag{9}$$

and $\Delta_{H_{-i}}$ denotes the determinant of the Bordered Hessian matrix of H_{-i}

$$\Delta_{H_{-i}} = \begin{vmatrix} 0 & H_1 & \cdots & H_{i-1} & H_{i+1} & \cdots & H_N \\ H_1 & H_{11} & \cdots & H_{1i-1} & H_{1i+1} & \cdots & H_{1N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{i-1} & H_{i-11} & \cdots & H_{i-1i-1} & H_{i-1i+1} & \cdots & H_{i-1N} \\ H_{i+1} & H_{i+11} & \cdots & H_{i+1i-1} & H_{i+1i+1} & \cdots & H_{i+1N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_N & H_{N1} & \cdots & H_{Ni-1} & H_{Ni+1} & \cdots & H_{NN} \end{vmatrix} \tag{10}$$

Since $H\left(q_1^{**},\dots,q_N^{**}\right)$ is strictly quasiconvex in q, we have $\partial q_i^{**}/\partial\theta_i > 0$. Thus $Q_i^{**}(\theta_i) = E_{\theta_{-i}} q_i^{**}(\theta_i,\theta_{-i})$ is also strictly increasing in θ_i (the monotonicity constraint holds). Summarizing the above analysis, we have

Proposition 1. (q^{**}, m^{**}) described above is a socially efficient spectrum sharing mechanism that satisfies IR, IC, and ITC.

Note that given $\partial q_i^{**}/\partial\theta_i > 0$ and (8), the efficient direct mechanism (q^{**}, m^{**}) is also dominant strategy incentive-compatible (Proposition 4.2, Börgers (2015)). This is similar to the traditional single-unit item allocation problem *a la* Myerson (1981), where the effcient mechanism can be implemented by a second-price sealed-bid auction (the Vickrey auction), which is a dominant-strategy IC mechanism. This resemblance is somewhat unexpected, as the complexity of additional noninterference constraints does not affect the dominant strategy implementation in our setting.

That $\partial q_i^{**}/\partial\theta_i > 0$ also implies

$$\frac{\partial m_i^{**}}{\partial\theta_i} = \theta_i \frac{\partial q_i^{**}}{\partial\theta_i} > 0.$$

This implies that $M_i^{**}(\theta_i) = E_{\theta_{-i}} m_i^{**}(\theta_i, \theta_{-i})$ is also strictly increasing in θ_i.

It turns out that the efficient mechanism can be implemented via an all-pay auction augmented by recovery functions $M_i^{**-1}(b_i)$, $i = 1, ..., N$ (which is made public before the sale). More specifically, bidders submit bids (if they so choose), and they need to pay what they bid regardless of the amount of power (q_i) being allocated. Given the bid profile b, a type profile θ will be recovered according to $\theta_i = M_i^{**-1}(b_i)$. The efficient power allocation rule will then be implemented accordingly.

Alternatively, we can consider a discriminatory-price auction augmented by the recovery function $\beta_i^{**-1}(b_i)$ where

$$\beta_i^{**}(\theta_i) = \theta_i - \frac{\int_{\underline{\theta}}^{\theta_i} Q_i^{**}(t_i)\,dt_i}{Q_i^{**}(\theta_i)}, \; i = 1, ..., N.$$

It works as follows: each bidder submits a bid (say, b_i), and will need to pay for the assigned power based on the unit-price equal to what she bids (those who are not assigned power do not pay, which is different from the all-pay auction counterpart). Given the bid profile b, a type profile θ will be recovered according to $\theta_i = \beta_i^{**-1}(b_i)$. The efficient power allocation rule will then be implemented accordingly.

Proposition 2. Socially efficient spectrum sharing allocation rule can be implemented in Bayesian Nash equilibrium via an all-pay auction or a discriminatory-price auction described in the preceding paragraphs.

Proof. See Appendix. □

In the proof, we basically show that bidding according to $M^{**}$and β^{**} constitutes Bayesian Nash equilibria in the all-pay auction and discriminatory-price auction, respectively, as described above. Note that an all-pay auction is quite robust in implementing desirable outcomes in different environments (for example, see Fullerton and McAfee, 1999, in the settings of tournaments, and Ye, 2007, and Lu and Ye, 2018, in the settings of two-stage auctions). But the implementation of the discriminatory-price auction is not guaranteed in many other environments (e.g., in the same settings of the tournaments and two-stage auctions mentioned above).

3.2 *Optimal Mechanisms*

We now turn to the revenue-maximizing (optimal) mechanisms. Let θ_i^* be the lowest possible type that user i would be willing to participate in the mechanism. By (4), we have

$$\theta_i Q_i(\theta_i) - M_i(\theta_i) = U_i\left(\theta_i^*\right) + \int_{\theta_i^*}^{\theta_i} Q_i(t_i)\,dt_i$$

which implies

$$M_i(\theta_i) = \theta_i Q_i(\theta_i) - \int_{\theta_i^*}^{\theta_i} Q_i(t_i)\,dt_i - U_i\left(\theta_i^*\right).$$

Following standard techniques in mechanism design, the expected payment from user i is given by

$$\begin{aligned} EM_i(\theta_i) &= \int_{\theta_i^*}^{\overline{\theta}} \int_{\theta_{-i}} \left\{ \theta_i Q_i(\theta_i) - \int_{\theta_i^*}^{\theta_i} Q_i(t_i)\,dt_i - U_i\left(\theta_i^*\right) \right\} f_{-i}(\theta_{-i})\,d\theta_{-i} f(\theta_i)\,d\theta_i \\ &= \int_{\theta_{-i}} \left\{ \int_{\theta_i^*}^{\overline{\theta}} \left[\theta_i Q_i(\theta_i) - \int_{\theta_i^*}^{\theta_i} Q_i(t_i)\,dt_i \right] f(\theta_i)\,d\theta_i \right\} f_{-i}(\theta_{-i})\,d\theta_{-i} - U_i\left(\theta_i^*\right) \\ &= \int_{\theta_{-i}} \left\{ \int_{\theta_i^*}^{\overline{\theta}} \left[\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i(\theta_i, \theta_{-i}) \right] f(\theta_i)\,d\theta_i \right\} f_{-i}(\theta_{-i})\,d\theta_{-i} - U_i\left(\theta_i^*\right). \\ &= \int_{\theta_i^*}^{\overline{\theta}} \int_{\theta_{-i}} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i})\,d\theta_{-i} f(\theta_i)\,d\theta_i - U_i\left(\theta_i^*\right). \end{aligned}$$

To maximize the expected revenue, the seller should set $U_i\left(\theta_i^*\right) = 0$ for all $i = 1, \ldots, N$. This also guarantees that IR is satisfied due to (4). The seller's objective therefore is reduced to find a mechanism that maximizes

$$\sum_{i=1}^{N} \int_{\theta_i^*}^{\overline{\theta}} \int_{\theta_{-i}} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) q_i(\theta_i, \theta_{-i}) f_{-i}(\theta_{-i})\,d\theta_{-i} f(\theta_i)\,d\theta_i$$

subject to the constraint that the mechanism is (1) IC: $Q_i'(\theta_i) \geq 0$ and (4), and (2) ITC: interference temperature constraint: (3).

Following similar procedures as in the analysis for efficient mechanisms, we can derive the following optimality condition:

$$\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} = \lambda(\theta) H_{q_i}(q_1, \ldots, q_N), \;\; i = 1, 2, \ldots, N, \tag{11}$$

where $\lambda(\theta)$ is the Lagrange parameter associated with the ITC. Condition (11) basically says that the optimal allocation rule equates marginal revenue (or *virtual value* a la Myerson) with marginal cost of power provision (the "cost" from the perspective of the principal).

Given the regularity condition and that $H_{q_i}(q_1, \ldots, q_N) > 0$, optimal power allocation involves market exclusion: all types below θ^* are not assigned power, where θ^* is the unique solution to $\theta - \frac{1-F(\theta)}{f(\theta)} = 0$. Note that θ^*, which is also known as the optimal screening threshold, is the same for all users in our model, as users are symmetric in terms of the value "types."

That revenue-maximizing allocation rule involves market exclusion is different from the efficient allocation rule, where market exclusion does not occur. Our result with the divisible good setting is consistent with the result from single-unit auctions (e.g., Myerson, 1981).

Combining the N equations (11) with (6), we can derive the optimal power allocation rule $q_i^* = q_i^*(\theta_1,...,\theta_N)$, $i = 1,2,...,N$.

Conditions (11) can be rewritten as

$$\frac{\theta_i - (1-F(\theta_i))/f(\theta_i)}{H_{q_i}\left(q_1^*,...,q_N^*\right)} = \frac{\theta_j - \left(1-F\left(\theta_j\right)\right)/f\left(\theta_j\right)}{H_{q_j}\left(q_1^*,...,q_N^*\right)} \text{ for } i,j = 1,...,N,\ i \neq j. \tag{12}$$

So unlike in the traditional optimal mechanism design settings where allocation favors the one with the highest virtual value, here the allocations favor those with the highest *relative virtual values*, which are adjusted by the cost due to the interference: the optimality requires that the ratio of marginal revenue over the marginal cost be equal across all users assigned positive units of power.

Following the same procedure as in the previous subsection, we can select the following payment rule:

$$m_i^*(\theta_1,...,\theta_N) = \theta_i q_i^*(\theta_1,...,\theta_N) - \int_{\theta^*}^{\theta_i} q_i^*(\theta_1,...,t_i,...,\theta_N)\,dt_i \tag{13}$$

where θ^* is the optimal screening level.

Since $H(q_1,...,q_N)$ is strictly quasi-convex, again we have

$$\frac{\partial q_i^*}{\partial \theta_i} = \frac{a(\theta_i)\Delta_{H_{-i}}}{\lambda \cdot \Delta_H} > 0,$$

where Δ_H is the determinant of the Bordered Hessian matrix of H and $\Delta_{H_{-i}}$ is the determinant of the Bordered Hessian matrix of H_{-i} defined in the previous subsection (but evaluated at $\left(q_i^*, q_{-i}^*\right)$ instead), and $a(\theta_i) = \partial[\theta_i - (1-F(\theta_i))/f(\theta_i)]/\partial\theta_i > 0$ (by Assumption 1). Following similar arguments for Proposition 1, we can establish

Proposition 3. (q^*, m^*) described above is a revenue-maximizing spectrum sharing mechanism that satisfies IR, IC, and ITC.

Again, given $\partial q_i^*/\partial\theta_i > 0$ and (13), the optimal direct mechanism (q^*, m^*) is also dominant strategy incentive-compatible (Proposition 4.2, Börgers, 2015). Note that in the traditional single-unit item allocation environment as studied by Myerson (1981), the optimal mechanism can be implemented by a second-price sealed-bid auction (the Vickrey auction) with appropriately set reserve prices, which is a dominant-strategy IC mechanism. Again, this resemblance is somewhat unexpected, given the additional noninterference constraints in our setting.

That $\partial q_i^*/\partial\theta_i > 0$ also implies that

$$\frac{\partial m_i^*}{\partial\theta_i} = \theta_i \frac{\partial q_i^*}{\partial\theta_i} > 0,$$

which in turn implies that $M_i^*(\theta_i) = E\left[m_i^*(\theta_i, \theta_{-i})\right]$ is strictly increasing in θ_i.

As in the case for efficient mechanism, the optimal mechanism can also be implemented via an all-pay auction or a discriminatory-price auction augmented by their corresponding recovery functions. Under an all-pay auction, M_i^{*-1} serves as the recovery function; under a discriminatory-price auction, β_i^{*-1} serves as the recovery function, where β_i^* is defined by

$$\beta_i^*(\theta_i) = \theta_i - \frac{\int_{\theta^*}^{\theta_i} Q_i^*(t_i)\,dt_i}{Q_i^*(\theta_i)}, \text{ for } \theta_i \geq \theta^*,\ i = 1, \ldots, N. \tag{14}$$

Again, after bids are collected, the underlying types are recovered based on the recovery function. The optimal power allocation rule can then be implemented accordingly.[8]

Proposition 4. The revenue-maximizing spectrum sharing allocation rule can be implemented in Bayesian Nash equilibrium via an all-pay auction described in the preceding paragraph.

Proof. The proof is basically the same as the proof for Proposition 2, with the only difference being that there is an optimal screening threshold at θ^* for the optimal mechanism. □

While an all-pay auction has an arguably undesirable feature that users need to pay even if they are not assigned any units of the good, this is not the case in our equilibrium: as long as a user submits a strictly positive bid, she will be assigned a positive amount of power in equilibrium.[9] This is due to Assumption 2. If H is not quasi-convex, it is possible that some users would not get assignments even if their value types are above θ^*.

It is also worth noting that under either an all-pay or a discriminatory-price auction in our implementations (both for efficiency and optimality), users follow asymmetric bid functions in equilibrium,

[8] Due to the optimal screening threshold, some users may not submit bids. So some underlying types may not be recovered. But those types must be below θ^* and hence would not affect the implementation of the optimal allocation rule.

[9] For those with types below the optimal screening threshold, they do not bid and hence do not pay in equilibrium.

although they are *ex ante* symmetric in terms of their type distributions. The reason is that despite symmetry in value types, they are typically asymmetric in terms of their externalities in causing interferences, which is reflected by the specific functional form of H. This also marks an interesting difference between our analysis and the traditional auction analysis where symmetry in value distributions usually leads to symmetric bid function in equilibrium.

3.3 An Example

Since remote sensing applications such as radar (e.g. air traffic control radar) are allocated primary use of a significant portion of the highly desirable spectrum below 6 GHz and typically have predictable patterns for spectral access that differs from standard communication applications such as LTE and Wi-Fi, spectrum scarcity issues encountered in wireless cellular and broadband have led to growing interest within both the research and regulatory communities to open radar spectrum for sharing with other applications.

Radar systems are usually narrow-band and seldom occupy their designated entire frequency band. Here multiple radar sites whose operating channels are interleaved across their allocated spectrum usually exploit conservative geographical separations to mitigate intra-system interference and provide the required coverage. From the perspective of a single radar site at any given time, much of its allotted spectrum lies idle due to the frequency and range separation, and hence representing a promising opportunity for non-harmful secondary transmission. The pulsed characteristics of radar signals together with the communications system processing chain also imply a general robustness of secondary transmissions against radar generated interference. With moderate frequency and range separation, spectrum sharing between a scanning ATC radar and secondary system is possible, although both of them may be ultimately subject to quantified performance degradations due to the impact each has on the other.

Below we present an example analysis of primary/secondary spectrum sharing for radar. The primary user (the observing system) is assumed to be an ATC radar working in L-Band. The secondary users are assumed to support the OFDM physical layer, which is a mainstream technology universally adopted in the most popular wireless cellular standard. Here we assume that both secondary users are Wi-Fi systems. The transmit power of the ith secondary system is denoted as p_i. The propagation loss between the transmitter of the ith secondary system and the receiver of the jth secondary system is denoted as h_{ij}. The power spectrum density of the receiver thermal noise is denoted as n_0, receiver bandwidth is denoted as B. The maximum allowable interference power at the radar receiver is denoted as P, which depends on the receiver sensitivity.

The receivers of secondary users are assumed to have receiver bandwidths set to be 160 MHz ($B = 1.6 \times 10^8$), which is a typical channel bandwidth specially designed for Wi-Fi systems. We also assume that the site planning is performed for the newly introduced secondary systems such that proper system isolation through range separation, frequency separation or phase encoding is achieved. It follows that the inter-system channel gain (h_{ij}, with $i \neq j$) is always smaller than the corresponding intra-system channel gain (h_{ij}, with $i = j$). More specifically, we assume $h_{11} = 2 \times 10^{-9}$, $h_{22} = 3 \times 10^{-9}$, and $h_{12} = h_{21} = 1 \times 10^{-11}$. The propagation loss between the primary system and the secondary systems takes typical values for LOS propagation with a range in tens of kilometers, and the propagation loss for secondary systems takes values for NLOS propagation with a range in hundreds of meters ($h_{10} = 3 \times 10^{-12}$, $h_{20} = 4 \times 10^{-12}$). We pick a value that is slightly larger than the receiver thermal noise power for the maximum tolerable interference at each receiver ($P = 10^{-12}$). In addition, we set $n_0 = 3.726 \times 10^{-21}$, which is a physical constant under room temperature. We also assume that the user type θ_i is uniformly distributed over $[0,1]$. Suppose the two secondary users' types are given by $\theta_1 = 0.6$, $\theta_2 = 0.8$, we will calculate the specific power allocation according to efficient and optimal mechanisms characterized in the preceding subsections.

First, the equation $\gamma_i = p_i h_{ii} / \left[n_0 B + \sum_{j \neq i} p_j h_{ji} \right]$, $i = 1,2,...,N$, can be rewritten as

$$\begin{pmatrix} \frac{h_{11}}{\gamma_1} & -h_{21} \\ -h_{12} & \frac{h_{22}}{\gamma_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = n_0 B \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we can solve for p_1 and p_2 in terms of γ_1 and γ_2:[10]

$$p_1 = \frac{n_0 B \left(\frac{h_{22}}{\gamma_2} + h_{21} \right)}{\frac{h_{11}}{\gamma_1} \frac{h_{22}}{\gamma_2} - h_{21} h_{12}}, \quad p_2 = \frac{n_0 B \left(\frac{h_{11}}{\gamma_1} + h_{12} \right)}{\frac{h_{11}}{\gamma_1} \frac{h_{22}}{\gamma_2} - h_{21} h_{12}}.$$

Since $\log(1 + \gamma_i) \equiv q_i$, $\gamma_i = \exp(q_i) - 1$, we can further write

$$p_1 = \frac{n_0 B \left(\frac{h_{22}}{\exp(q_2) - 1} + h_{21} \right)}{\frac{h_{11}}{\exp(q_1) - 1} \frac{h_{22}}{\exp(q_2) - 1} - h_{21} h_{12}}, \quad p_2 = \frac{n_0 B \left(\frac{h_{11}}{\exp(q_1) - 1} + h_{12} \right)}{\frac{h_{11}}{\exp(q_1) - 1} \frac{h_{22}}{\exp(q_2) - 1} - h_{21} h_{12}}$$

[10]The condition required is

$$\begin{vmatrix} \frac{h_{11}}{\gamma_1} & -h_{21} \\ -h_{12} & \frac{h_{22}}{\gamma_2} \end{vmatrix} \neq 0.$$

The constraint $\sum_{i=1}^{N} p_i h_{i0} \le P$ can now be rewritten as

$$\frac{n_0 B\left(\frac{h_{22}}{\exp(q_2)-1}+h_{21}\right)}{\frac{h_{11}}{\exp(q_1)-1}\frac{h_{22}}{\exp(q_2)-1}-h_{21}h_{12}}h_{10}+\frac{n_0 B\left(\frac{h_{11}}{\exp(q_1)-1}+h_{12}\right)}{\frac{h_{11}}{\exp(q_1)-1}\frac{h_{22}}{\exp(q_2)-1}-h_{21}h_{12}}h_{20}=H(q_1,q_2)\le P$$

1. Efficient allocation. Solving from

$$\begin{aligned}\frac{\theta_1}{H_{q_1}(q_1,q_2)} &= \frac{\theta_2}{H_{q_2}(q_1,q_2)}\\ H(q_1,q_2) &= P\end{aligned}$$

we have

$$\begin{aligned} q_1^{**} &= 4.2600,\ q_2^{**}=5.9523\\ p_1^{**} &= 0.0856,\ p_2^{**}=0.1857\end{aligned}$$

2. Optimal allocation. Solving from

$$\begin{aligned}\frac{\theta_1-(1-F(\theta_1))/f(\theta_1)}{H_{q_1}(q_1,q_2)} &= \frac{\theta_2-(1-F(\theta_2))/f(\theta_2)}{H_{q_2}(q_1,q_2)}\\ H(q_1,q_2) &= P\end{aligned}$$

we have

$$\begin{aligned} q_1^{**} &= 2.7312,\ q_2^{**}=6.7703\\ p_1^{**} &= 0.0211,\ p_2^{**}=0.2342\end{aligned}$$

4 Extensions

In the preceding analysis we assume that the distribution of the users' types satisfies a regularity condition (Assumption 1). Moreover, we assume deterministic channel gain structure (h_{ij}'s are deterministic). In this section we demonstrate how we can extend our analysis to the settings where either of these assumptions is relaxed.

4.1 Analysis With General Type Distributions

Assumption 1 ensures that the monotonicity constraint $Q_i'(\theta_i) \geq 0$ is nonbinding. When Assumption 1 fails, it is possible that the monotonicity constraint can be binding and the solution may involve bunching (i.e., there exist intervals over which the allocation of power is constant). Taking into account the monotonicity constraint explicitly and letting $w_i(\theta_i,\theta_{-i}) = \partial q_i(\theta_i,\theta_{-i})/\partial\theta_i$, the seller's revenue-maximization problem is as follows:

$$\max_{(q_1,\ldots,q_N)} \sum_{i=1}^{N} \int_{\theta_i^*}^{\overline{\theta}} \int_{\theta_{-i}} \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) q_i(\theta_i,\theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i} f(\theta_i) d\theta_i$$

Subject to

$$\begin{aligned} Q_i'(\theta_i) = \int_{\theta_{-i}} \frac{\partial q_i(\theta_i,\theta_{-i})}{\partial\theta_i} f_{-i}(\theta_{-i}) d\theta_{-i} &= \int_{\theta_{-i}} w_i(\theta_i,\theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i} \\ \int_{\theta_{-i}} w_i(\theta_i,\theta_{-i}) f_{-i}(\theta_{-i}) d\theta_{-i} &\geq 0 \\ H(q_1(\theta_1,\ldots,\theta_N),\ldots,q_N(\theta_1,\ldots,\theta_N)) &\leq P \end{aligned}$$

The Hamiltonian for this program is

$$\mathscr{H}(\theta_i, q_i(\theta_i,\theta_{-i}), w_i(\theta_i,\theta_{-i}), \lambda_i(\theta_i), \lambda(\theta)) = \left\{ \begin{array}{c} \sum_{i=1}^{N} \left[\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) q_i(\theta_i,\theta_{-i}) + \lambda_i(\theta_i) w_i(\theta_i,\theta_{-i}) \right] \\ +\lambda(\theta)[P - H(q_1(\theta_1,\ldots,\theta_N),\ldots,q_N(\theta_1,\ldots,\theta_N))] \end{array} \right\}$$

By Pontryagin's maximum principle, the necessary conditions for an optimum $\left[\overline{q}_i(\theta_i,\theta_{-i}), \overline{w}_i(\theta_i,\theta_{-i})\right]$ are given by

1.

$$\begin{aligned} & \mathscr{H}\left(\theta_i, \overline{q}_i(\theta_i,\theta_{-i}), \overline{w}_i(\theta_i,\theta_{-i}), \overline{q}_{-i}(\theta_i,\theta_{-i}), \overline{w}_{-i}(\theta_i,\theta_{-i}), \lambda_i(\theta_i), \lambda(\theta)\right) \\ \geq\ & \mathscr{H}\left(\theta_i, \overline{q}_i(\theta_i,\theta_{-i}), w_i(\theta_i,\theta_{-i}), \overline{q}_{-i}(\theta_i,\theta_{-i}), \overline{w}_{-i}(\theta_i,\theta_{-i}), \lambda_i(\theta), \lambda(\theta)\right) \end{aligned}$$

2. Except at points of discontinuity of $\overline{q}_i(\theta_i,\theta_{-i})$, we have

$$\frac{d\lambda_i(\theta)}{d\theta_i} = -\left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,\ldots,\theta_N),\ldots,\overline{q}_N(\theta_1,\ldots,\theta_N)\right)\right], \ i = 1,\ldots,N. \qquad (15)$$

3. The transversality conditions $\lambda_i\left(\underline{\theta}\right) = \lambda_i\left(\overline{\theta}\right) = 0$ are satisfied.

Integrating equation(15), we have

$$\lambda_i(\theta_i) = \int_{\theta_i}^{\overline{\theta}} \left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,\ldots,\theta_N),\ldots,\overline{q}_N(\theta_1,\ldots,\theta_N)\right)\right] d\theta_i.$$

The first condition above requires that $\overline{w}_i(\theta_i,\theta_{-i})$ maximize

$$\mathcal{H}\left(\theta_i, \overline{q}_i(\theta_i,\theta_{-i}), w_i(\theta_i,\theta_{-i}), \overline{q}_{-i}(\theta_i,\theta_{-i}), \overline{w}_{-i}(\theta_i,\theta_{-i}), \lambda_i(\theta), \lambda(\theta)\right)$$

subject to $w_i(\theta_i,\theta_{-i}) \geq 0$. This requirement implies $\lambda_i(\theta_i) \leq 0$, or

$$\int_{\theta_i}^{\overline{\theta}} \left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,\ldots,\theta_N),\ldots,\overline{q}_N(\theta_1,\ldots,\theta_N)\right)\right] d\theta_i \leq 0.$$

Whenever $\lambda_i(\theta) < 0$ we must have

$$\overline{w}_i(\theta_i,\theta_{-i}) = \frac{\partial \overline{q}_i(\theta_i,\theta_{-i})}{\partial \theta_i} = 0.$$

Thus we have the following complementary slackness condition:

$$\frac{\partial \overline{q}_i(\theta_i,\theta_{-i})}{\partial \theta_i} \cdot \int_{\theta_i}^{\overline{\theta}} \left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} - \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,\ldots,\theta_N),\ldots,\overline{q}_N(\theta_1,\ldots,\theta_N)\right)\right] d\theta_i = 0$$

for all $\theta_i \in \left[\underline{\theta},\overline{\theta}\right]$.

It follows from this condition that if $\overline{q}_i(\theta_i,\theta_{-i})$ is strictly increasing over some interval, then it must coincide with q_i^s, the solution that would have been derived following the procedure introduced in Section 3.2. To see this conclusion, note that $\overline{w}_i(\theta_i,\theta_{-i}) = \partial \overline{q}_i(\theta_i,\theta_{-i})/\partial \theta_i > 0$, hence $\lambda_i(\theta_i) = 0$ and $d\lambda_i(\theta_i)/d\theta_i = 0$, which implies

$$\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} = \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,\ldots,\theta_N),\ldots,\overline{q}_N(\theta_1,\ldots,\theta_N)\right).$$

But this is precisely the condition that defines q_i^s. It therefore only remains to determine the bunching intervals over which $\overline{q}_i(\theta_i,\theta_{-i})$ is constant. Suppose $\left[\theta^1,\theta^2\right]$ is such an interval.

Consider Figure 2. To the left of θ^1 and to the right of θ^2, we have

$$\lambda_i(\theta_i) = 0 \text{ and } \overline{w}_i(\theta_i,\theta_{-i}) = \frac{\partial \overline{q}_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \frac{\partial q_i^s(\theta_i,\theta_{-i})}{\partial \theta_i} > 0.$$

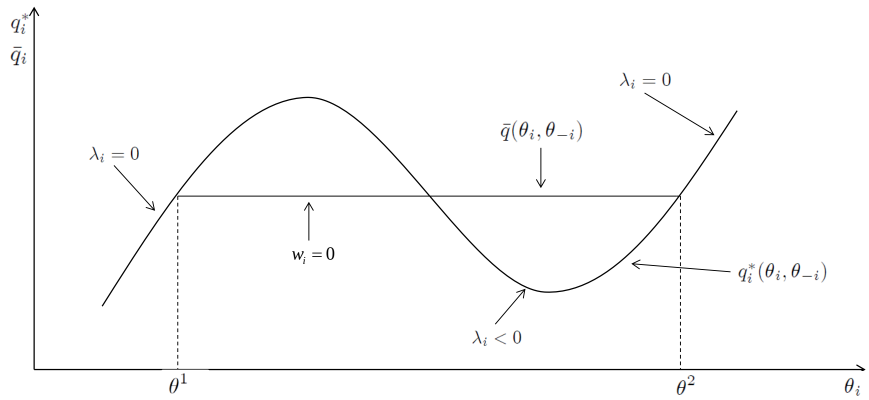

Figure 2: An Illustration of Bunching

For any θ between $\theta^1$and θ^2, we have

$$\lambda_i(\theta_i) < 0 \text{ and } \overline{w}_i(\theta_i, \theta_{-i}) = 0.$$

By continuity of $\lambda_i(\theta_i)$, we must have $\lambda_i(\theta^1) = \lambda_i(\theta^2) = 0$, so that

$$\int_{\theta^1}^{\theta^2} \left[\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1, \ldots, \theta_N), \ldots, \overline{q}_N(\theta_1, \ldots, \theta_N)\right) \right] d\theta = 0. \tag{16}$$

In addition, at $\theta^1$and θ^2 we must have $q_i^s(\theta^1, \theta_{-i}) = q_i^s(\theta^2, \theta_{-i})$. This follows from the continuity of $\overline{q}_i(\theta_i, \theta_{-i})$. Thus we have two additional equations, allowing us to determine the two additional unknowns $\theta^1$and θ^2.

Given $(\theta_1, \ldots, \theta_N)$, suppose $\partial\overline{q}_i(\theta_i, \theta_{-i})/\partial\theta_i = 0$ for $\theta_i \in [\theta^1, \theta^2]$. We have

$$\overline{q}_i(\theta_i, \theta_{-i}) = q_i^s(\theta^1, \theta_{-i})$$

for $\theta_i \in [\theta^1, \theta^2]$. Suppose $\partial\overline{q}_j(\theta_j, \theta_{-j})/\partial\theta_j = \partial q_j^s(\theta_j, \theta_{-j})/\partial\theta_j > 0$ for $j \neq i$. Then we have

$$\theta_j - \frac{1-F(\theta_j)}{f(\theta_j)} = \lambda(\theta) H_{q_j}\left(\overline{q}_1(\theta_1,...,\theta_N),...,\overline{q}_N(\theta_1,...,\theta_N)\right)$$

for $j \neq i$.

We also have

$$H\left(\overline{q}_1(\theta_1,...,\theta_N),...,\overline{q}_N(\theta_1,...,\theta_N)\right) = P$$

Combining these $N+1$ equations, we can solve for all $\overline{q}_i(\theta_1,...,\theta_N)$ and $\lambda(\theta_1,...,\theta_N)$. It is easily verified that

$$\frac{\partial \overline{q}_i(\theta_1,...,\theta_N)}{\partial \theta_i} = \frac{\partial \overline{q}_j(\theta_1,...,\theta_N)}{\partial \theta_i} = \frac{\partial \lambda(\theta_1,...,\theta_N)}{\partial \theta_i} = 0, \text{ for } \theta_i \in \left[\theta^1, \theta^2\right].$$

So equation (16) leads to

$$\int_{\theta^1}^{\theta^2} \left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right] d\theta_i / \left(\theta^2 - \theta^1\right) = \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,...,\theta_N),...,\overline{q}_N(\theta_1,...,\theta_N)\right).$$

At $\theta^1$and θ^2, we have $q_i^s\left(\theta^1,\theta_{-i}\right) = q_i^s\left(\theta^2,\theta_{-i}\right) = \overline{q}_i(\theta_1,...,\theta_N)$ for $\theta_i \in \left[\theta^1,\theta^2\right]$ (by the continuity of $\overline{q}_i(\theta_i,\theta_{-i})$). So we have

$$\theta^1 - \frac{1-F\left(\theta^1\right)}{f\left(\theta^1\right)} = \int_{\theta^1}^{\theta^2} \left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right] d\theta_i / \left(\theta^2 - \theta^1\right) = \theta^2 - \frac{1-F\left(\theta^2\right)}{f\left(\theta^2\right)} \tag{17}$$

Define the virtual value or marginal revenue $v(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$. In the case when the regularity condition holds, we have derived the following optimality condition in Section 3.2:

$$v(\theta_i) = \lambda(\theta) H_{q_i}(q_1,...,q_N), \ i = 1,2,...,N,$$

In the general case allowing for general type distributions, $v(\theta)$ may not always increase in θ. We need to iron $v(\theta)$ to get $\overline{v}(\theta)$ following the procedure described above. When $v(\theta)$ decreases in θ over some intervals, we can use (17) to determine bunching endpoints $\theta^1$and θ^2. Let

$$\begin{aligned} \overline{v}(\theta) &= \theta^1 - \frac{1-F\left(\theta^1\right)}{f\left(\theta^1\right)}, \text{ if } \theta \in \left[\theta^1, \theta^2\right] \\ \overline{v}(\theta) &= \theta - \frac{1-F(\theta)}{f(\theta)}, \text{ otherwise} \end{aligned}$$

Now we can derive the following optimality condition for the general case:

$$\overline{v}(\theta_i) = \lambda(\theta) H_{q_i}\left(\overline{q}_1(\theta_1,...,\theta_N),...,\overline{q}_N(\theta_1,...,\theta_N)\right),\ i = 1,2,...,N, \tag{18}$$

Combining the N equations (18) with the binding ITC constraint, we can derive the optimal allocation rule $\overline{q}$. Once $\overline{q}$ is identified, we can follow similar procedures as described in Section 3. 2 to derive the payment rule $\overline{m}$. Proposition 3 can then be extended to accommodate the case with general type distribution:

Proposition 5. $(\overline{q},\overline{m})$ is a revenue-maximizing spectrum sharing mechanism that satisfies IR, IC, and ITC in the case with general type distribution.

Intuitively, when the regularity condition fails, the optimal allocation rule needs to be modified as follows: the optimality requires that the ratio of *average* marginal revenue over the marginal cost be equal across all users assigned positive units of power.

4.2 The Stochastic Channel Gains

Unlike in the preceding analysis, we now assume that h_{ij}'s and h_{i0}'s are stochastic. Let h_{ij} be the s-tochastic channel gain from user i's transmitter to user j's receiver, and h_{i0} be the stochastic channel gain from user i's transmitter to the measurement point. Ex ante, we assume that h_{ij}'s are independently distributed according to distribution function $G_{ij}(\cdot)$ over support $[\underline{h},\overline{h}]$, and h_{i0}'s are independently distributed according to distribution function $G_{i0}(\cdot)$ over support $[\underline{h},\overline{h}]$.

Given h_{ij}'s and h_{i0}'s, the received Signal-to-Interference plus Noise Ratio (SINR) is again given by

$$\gamma_i = \frac{p_i h_{ii}}{n_0 B + \sum_{j\neq i} p_j h_{ji}},\ \ i = 1,2,...,N. \tag{19}$$

The interference temperature constraint, however, now requires that the expected total received power at a specified measurement point satisfy the following condition:

$$\sum_{i=1}^{N} p_i \int_{\underline{h}}^{\overline{h}} h_{i0} dG_{i0}(h_{i0}) \leq P. \tag{20}$$

Given type θ_i and γ_i, user i's valuation of the spectrum is now given by the expected utility $\theta_i E\left(\log(1+\gamma_i)\right)$, where $E\left(\log(1+\gamma_i)\right) \equiv q_i$ is a measure for the expected desired Quality of Service (QoS). As before, under certain regular conditions, $\{p_i\}_{i=1}^N$ can be uniquely determined by $\{q_i\}_{i=1}^N$. So the interference temperature constraint (2) can be reformulated in terms of q_i's:

$$H(q_1,...,q_N) \le P.$$

We can then define the QoS provision possibility set:

$$\mathbf{Q} = \{(q_1,...,q_N) : H(q_1,...,q_N) \le P\}.$$

Starting from here, the analysis can be carried out as in our main model.

5 Conclusion

In this paper we study socially efficient and revenue-maximizing mechanisms in a special radio spectrum sharing environment. The mechanism design problem in such a setting is different from the traditional setting mainly in that we need to explicitly take into account a non-interference constraint. Incorporating such a constraint may potentially complicate the analysis; nevertheless under some regularity conditions we are able to fully characterize both the socially efficient and revenue optimal mechanisms. Unlike in the traditional mechanism design literature where the allocation is determined by the values or virtual values alone, in our setting the allocation depends on *relative* values or virtual values, which are adjusted by a term reflecting the marginal cost in terms of the externality caused to the interference. The efficient and optimal allocation rules identified thus have intuitive economic interpretations. We show that both efficient and optimal mechanisms are also dominant strategy incentive-compatible, and that they can both be implemented via some well-designed all-pay or discriminatory-price auctions equipped with recovery functions.

Our model is stylized, but can be generalized along some directions (as discussed in Section 4). Another important extension is to allow for multi-dimensional technological constraints implied by spectrum sharing. For example, in the radar application, restrictions in time, frequency, and location impose multiple constraints, and they should all be taken into account in designing a viable market for spectrum sharing (Johnson et al., 2014). A direct implication is that optimal mechanisms involve allocation of not only power but also frequency bands or time intervals, which is more challenging than the case analyzed in this current paper. Despite the complications, we believe that some general insights obtained from this current model should still be robust. In particular, we expect that for the revenue-maximizing mechanisms, the principal should still maximize a generalized virtual value adjusted by all the constraints, and the power or bandwidth will be allocated to the extent that each recipient will contribute to the same ratio of marginal virtual value over marginal cost (in terms of the negative externalities caused to the interference). Given the technical difficulties, this is left for future research.

APPENDIX

Proof of Proposition 2: To show that the efficient allocation rule can be implemented by the described all-pay auction (with a trivial reserve price at zero), it only remains to show that it is a Bayesian Nash equilibrium (BNE) for users to bid according to $\left(M_1^{**},...,M_N^{**}\right)$.

Given that everyone else bids according to $M_{-i}^{**}(\cdot)$, bidder i maximizes the following objective function by choosing her bid b:

$$\widehat{\Pi}(b,\theta_i) = E_{\theta_{-i}}\left[\theta_i q_i^{**}(M_i^{**-1}(b),\theta_{-i}))\right] - b = \theta_i Q_i^{**}(M_i^{**-1}(b)) - b$$

We next apply the constraint simplification theorem[11] to demonstrate that bidding according to $\left(M_1^{**},...,M_N^{**}\right)$ constitutes a BNE in this all-pay auction game. This can be verified in the following steps:

1) $M_i^{**}(\cdot)$ is strictly increasing as shown in the text.

2) Since $Q_i^{**\prime}(\cdot) > 0$, $\widehat{\Pi}(b,\theta_i)$ satisfies the strict and smooth single crossing differences property.

3) By construction of M_i^{**}, it verifies the envelope formula.

4) It is also easily verified that bidding outside the range of $M_i^{**}(\cdot)$ cannot lead to higher expected payoff.

Thus all the sufficiency conditions for the constraint simplification theorem are satisfied and $M^{**} = \left(M_1^{**},...,M_N^{**}\right)$ indeed constitutes an BNE in the described all-pay auction game.

The implementation via the described discriminatory-price auction can be demonstrated analogously. It now remains to show that it is an BNE for users to bid according to $\left(\beta_1^{**},...,\beta_N^{**}\right)$, where β_i^{**} is given by (14).

Given that everyone else bids according to $\beta_{-i}^{**}(\cdot)$, bidder i maximizes the following objective function

[11] See, for example, Theorem 4.3 in Milgrom [13], pp. 105.

by choosing her bid b:

$$\widehat{\Pi}(b,\theta_i) = E_{\theta_{-i}}\left[(\theta_i - b)q_i^{**}(\beta_i^{**-1}(b),\theta_{-i})\right] = (\theta_i - b)Q_i^{**}(\beta_i^{**-1}(b))$$

1) It is easily verified that $\beta_i^{**}(\cdot)$ is strictly increasing.

2) Since $Q_i^{**\prime}(\cdot) > 0$, $\widehat{\Pi}(b,\theta_i)$ satisfies the strict and smooth single crossing differences property.

3) β_i^{**} verifies the envelope formula: $\widehat{\Pi}(\beta_i^{**}(\theta_i),\theta_i) = (\theta_i - \beta_i^{**}(\theta_i))Q_i^{**}(\theta_i) = \int_{\underline{\theta}}^{\theta_i} Q_i^{**}(t_i)\,dt_i$.

4) It is also easily verified that bidding outside the range of $\beta_i^{**}(\cdot)$ cannot lead to higher expected payoff.

Thus all the sufficiency conditions for the constraint simplification theorem are satisfied and β^{**} indeed constitutes an BNE in the described discriminatory-price auction game.

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