Auctions with Synergy and Resale

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Abstract

We study a sequential auction of two objects with two bidders, where the winner of the package obtains a synergy from the second object. If reselling after the two auctions takes place, it proceeds as either a monopoly or a monopsony, that makes a take-it-or-leave-it offer. We find that a post-auction resale has a significant impact on bidding strategies in the auctions: Under the monopoly offer, there does not exist an equilibrium (symmetric or asymmetric) where bidders reveal their types with positive probability. Under the monopsony offer, however, we can identify symmetric increasing equilibrium strategies in auctions for both items. While allowing resale always improves efficiency, we demonstrate that the effect of resale on expected revenue and the probability of exposure are both ambiguous.

Keywords: Auctions, auctions with synergy, resale

JEL Classification: D44, D80, D82, D40

1 Introduction

In auctions with multiple objects, obtaining more than one object often makes the value of a package larger than the sum of all the stand-alone values. This superadditivity is referred to as *synergy* in the auction literature. Synergies can arise due to a variety of reasons: they can be due to the complementarity derived from a subset of objects auctioned, the geographic advantage in cost saving

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among nationwide companies, or skills learning from previous objects, etc.[1]

Mainly motivated by the spectrum auctions conducted all over the world in the last decade, there is a growing literature on auctions with synergies. Rosenthal and Wang (1996) study a simultaneous-auction model with synergies and common values. They construct strategies in which bidders of different types randomize over different bid intervals and provide conditions for such strategies to form symmetric equilibria. Krishna and Rosenthal (1996) find that global bidders (e.g., nationwide companies) bid less aggressively than local bidders when there are more competitors. However, Rothkopf et al. (1998) argue that though attractive, package bidding may create new problems such as the huge complexity in determining the optimal bids for the most interested package when the objects auctioned are numerous. Ausbel and Milgrom (2002) propose proxy bidding in ascending package auctions and show that even when goods are complementary, ascending proxy auction equilibria lie in the core with respect to the true preferences. Kagel and Levin (2005) experimentally compare open outcry and sealed-bid uniform-price auctions with synergies. They identify two forces behind the bidding behavior, one due to demand reduction and the other due to the "exposure problem" (bidders may win only part of a package and earn negative profits). Their results show that bidding outcomes are closer to equilibrium in clock compared to sealed-bid auctions. Cantillon and Pesendorfer (2006) conduct an empirical analysis on London bus route auctions and conclude that the benefits of allowing package bidding is ambiguous.

Synergies also arise in sequential auctions. For example, in the procurement context, large projects are usually divided into subprojects and auctioned sequentially. When similar projects are auctioned within a relatively short period of time (compared to the time length to complete the project), it is natural to conjecture that by winning multiple projects, companies will save some cost either because of savings in fixed costs or improvements in techniques by learning. Branco (1997) modified Krishna and Rosenthal (1996) into a sequential auction of two identical items to a global bidder and two local bidders, where one local bidder only wants the first item, while the other only wants the second. In this setting he identifies a decline in the price trend. Menezes and Monteiro (2003, 2004) modify Branco (1997) in that local bidders (with a single demand) have equal valuations for both items. They show that with this modification, bidding price may not necessarily

[1] For example, Ausubel et al. (1997) provide some evidence for synergies in wireless telephone from the broadband PCS auctions.

decline because local bidders bid less aggressively in the first auction. Jeitshko and Wolfstetter (2002) analyze a model with two bidders and two non-identical items in a sequential setting. Evaluations for both items are drawn from a Bernoulli distribution with the same support. However, the synergy is introduced in that the probability to get a higher valuation for the second item is higher for the winner of the first item. They show that positive synergy intensifies competition in the first auction and causes a decline in the price trend.

While a common approach to modeling synergy is to add a constant value to the sum of the stand-alone values of the package, Menezes and Monteiro (2003, 2004) warn that it is unrealistic to consider that a package of relatively small value will yield the same synergy as a package with a relatively high value. Leufkens et al. (2006, 2007) consider both theoretically and experimentally a sequential auction of two projects where the synergy factor appears as a multiplier on the stand-alone value of the second project. They model the sequential auctions as second-price sealed bid, and show that the presence of synergies induces more competitive bidding, leading to lower expected profit to the bidders and higher expected revenue for the seller. The presence of synergies also leads to a decreasing price trend.

The contribution of this paper is to model resale in the framework adopted by Leufkens et al. (2006, 2007) described above. As is well known, one undesirable feature caused by synergies (in both simultaneous and sequential auctions) is that the auction outcome is typically inefficient (three types of inefficient outcomes are identified by Leufkens et al., 2006). A natural way to restore efficiency is to allow for post-auction resale. However, as the previous literature on resale has pointed out, the possibility of resale changes bidder behavior substantially, and the existence of a monotonic equilibrium is nontrivial. Haile (2003) considers a setup in which bidders only have noisy signals at the auction stage, where the motive for resale arises when the true value of the auction winner turns out to be low. Garret and Tröger (2006) consider a model with a speculator, who can only benefit from participating in the auction when she can resell the item to the other bidder. Following that, Garret et al. (2008) prove the existence of a seemingly collusive equilibrium in a standard English auction with resale, where this equilibrium Pareto dominates the "bid your value" equilibrium. Hafalir and Krishna (2008) analyze auctions with resale in an asymmetric independent private value auction environment and show that the expected revenue is higher under a first-price auction than under a second-price auction.

By explicitly taking into account post-auction resale, we demonstrate that the specific resale mechanism matters for the existence of an equilibrium in the bidding stage. More specifically, if the post-auction resale takes the form of a monopoly offering (i.e., the seller makes the take-it-or-leave-it offer), we fail to identify any equilibrium (symmetric or asymmetric) in which a bidder has to reveal her type over some range of values with positive probability. When the resale takes the form of a monopsony offering (i.e., the buyer makes the take-it-or-leave-it offer), however, we show that a symmetric increasing equilibrium exists. In the case with a monopsony offer, we show that while bidders bid higher for the second item when resale is available, the effect of resale on the bidding in the first auction is ambiguous. Thus, while allocation efficiency always improves with resale, the expected revenue to the seller can either increase or decrease with resale. Moreover, we demonstrate that the "exposure" problem (i.e., over-paying for one item without winning both items) can be either more severe or less severe when resale is allowed; if the synergy effect is sufficiently large, however, allowing resale will unambiguously lead to a lower probability of exposure.

This paper is organized as follows. Section 2 introduces the model; Section 3 considers the monopoly offering mechanism in resale; Section 4 considers the monopsony offering mechanism in resale; Section 5 concludes.

2 The Model

We consider a private and independent value auction with two risk neutral bidders. Two items are auctioned sequentially using the second-price sealed bid format. The stand-alone value of the first item (x_i) is distributed according to the differentiable CDF $F(\cdot)$ with a positive density function $f(\cdot)$ over $[0,1]$. The stand-alone value of the second item (y_i) follows a Bernoulli distribution, taking value 1 (High) with probability $p \in (0,1)$, and 0 (Low) with probability $1-p$. If a bidder only obtains a single item, the value is the stand-alone value; if a bidder obtains both items, the value is $x_i + \theta y_i$, where the synergy factor $\theta \in (1, +\infty)$.

We consider post-auction resale so that after the completion of both auctions, resale may take place in the form of either a monopoly offer or a monopsony offer. In a monopoly mechanism, the seller, who won at least one item from the auctions, makes a take-it-or-leave-it offer to the other bidder; in a monopsony mechanism, the buyer, who won at most one item from the auctions and wants to buy one or two items from the other bidder, makes a take-it-or-leave-it offer to the other bidder. Without loss of generality, we assume that offers are accepted whenever the other bidder is weakly better off by taking the offer.

As is standard in auction analysis, we assume that ties are announced and are broken at random. For notational convenience, we refer to winner 1/ loser 1 as the winner/loser of the first item.

When resale is absent or banned, this model is a special case in Leufkens et al. (2006). It is easily verified that the symmetric equilibrium for the first auction is given by

$$\beta(x_i) = x_i + p(\theta - 1).$$

Note that bidders only bid their values when there is no synergy between the two items for sale (i.e., when $\theta = 1$).

In the second auction, winner 1 and loser 1's bidding strategies are different because of the synergy: in equilibrium, winner 1 bids θy_i and loser 1 bids y_i.

In the symmetric equilibrium described above, it can be easily seen that when loser 1 obtains signal 1, while winner 1 obtains signal 0, the outcome is not efficient when resale is banned. Both bidders can be better off if winner 1 can resell the first item to loser 1. However, when resale is allowed, the option value and additional profit induced by resale will affect bidding strategies in the first auction. Moreover, the resale mechanism matters for the final allocation. In this paper, we analyze two mechanisms in the resale stage: the monopoly case, where the seller of the item makes a take-it-or-leave-it offer, and the monopsony case, where the buyer makes a take-it-or-leave-it offer.

3 Resale with Monopoly Offer

We consider in this section that the bidder who wins at least the first item makes the offer in the resale stage since it is obvious that if one bidder only won the second item, she does not strictly benefit from reselling it to the other bidder. We will rule out any equilibrium in which at least one bidder reveals her first-auction value over some non-degenerate interval. To that end, we start with the assumption that there exist a pair of equilibrium bidding strategies in the first auction which are represented by $\beta_i(\cdot) : [0, 1] \to \mathbf{R}_+$, $i = 1, 2$. Suppose $\beta_i(x_i)$ is a piecewise continuous function over the disjoint intervals $[x_{i,n-1}, x_{i,n})$, $n = 1, 2, ..., N - 1$, and $[x_{i,N-1}, 1]$, where $x_{i,0} = 0$; and that

$\beta_i(x_i)$ is weakly increasing over $[0,1]$.[2] Without loss of generality, we assume that $\beta_1(\cdot)$ is strictly increasing over a non-degenerated interval $(\underline{v}_1, \bar{v}_1)$ (so that bidder 1 reveals her first-auction signal over this interval in equilibrium) and $\beta_1(\bar{v}_1) \leq \beta_2(1)$ (so that bidder 1 does not always win the first item).

The following lemma establishes that there exists an interval $(\underline{v}_2, \bar{v}_2)$ over which $\beta_2(\cdot)$ is strictly increasing and that $\beta_1(\underline{v}_1, \bar{v}_1)$ and $\beta_2(\underline{v}_2, \bar{v}_2)$ have some non-degenerated overlap, where $\beta_i(\underline{v}_i, \bar{v}_i) = (\beta_i(\underline{v}_i), \beta_i(\bar{v}_i))$.

Lemma 1 *There exists $(\underline{b}_1, \bar{b}_1) \subset \beta_1(\underline{v}_1, \bar{v}_1)$, such that the interval $\beta_2^{-1}(\underline{b}_1, \bar{b}_1)$ is non-degenerated and that $\beta_2(\cdot)$ is strictly increasing over this interval.*

Proof. Suppose not. Then, either bidder 2 never bids over the interval $\beta_1(\underline{v}_1, \bar{v}_1)$, or all types of bidder 2 who bid over this interval are pooling or partial pooling (i.e., this part of the bidding function is a step-wise function). In the first case, instead of following a strictly increasing bid function over $(\underline{v}_1, \bar{v}_1)$, bidder 1 will be better off by pooling at $\beta_1(\underline{v}_1)$, because doing so does not switch her from winning to losing or vice versa but she benefits from hiding her first item value from the potential seller during resale. In the second case, given that bidder 2 can only pool at a finite number of bids, we can identify subintervals in $\beta_1(\underline{v}_1, \bar{v}_1)$ over which bidder 2 never bids on. A similar argument from the first case can be applied to reach a contradiction. ■

Given Lemma 1, we can find a subinterval $(\underline{x}_1, \bar{x}_1)$ in $(\underline{v}_1, \bar{v}_1)$ and that $\beta_2(\cdot)$ is strictly increasing over $\beta_2^{-1}(\beta_1(\underline{x}_1), \beta_1(\bar{x}_1)) \equiv (\underline{x}_2, \bar{x}_2)$.

Lemma 2 *There exists intervals $(x_{1*}, x^{1*}) \subset (\underline{x}_1, \bar{x}_1)$ and $(x_{2*}, x^{2*}) \subset (\underline{x}_2, \bar{x}_2)$ such that when bidder 1 with $x_1 \in (x_{1*}, x^{1*})$, or bidder 2 with $x_2 \in (x_{2*}, x^{2*})$ loses, resale occurs with positive probability.*

Proof. We examine all possibilities in order:

1. $\bar{x}_1 \leq \underline{x}_2$. In this case, if bidder 2 with $x_2 \in (\underline{x}_2, \bar{x}_2)$ loses, she knows that $x_2 < x_1$ with positive probability; thus she expects that resale will occur with positive probability.

[2] We will focus on the candidate equilibria where both bidding functions are weakly increasing, but our proof should apply to the case in which one bidder follows a weakly increasing bidding function while the other bidder follows a weakly decreasing bidding function.

2. $\underline{x}_1 \geq \bar{x}_2$. Similarly to the case above, if bidder 1 with $x_1 \in (\underline{x}_1, \bar{x}_1)$ loses, resale will occur with positive probability.

3. $(\underline{x}_1, \bar{x}_1)$ overlaps $(\underline{x}_2, \bar{x}_2)$. Without loss of generality, assume that $\bar{x}_2 \leq \bar{x}_1$.

   (a) $\underline{x}_1 = \underline{x}_2, \bar{x}_1 = \bar{x}_2$, and $\beta_1(x) = \beta_2(x)$ for all $x \in (\underline{x}_1, \bar{x}_1)$. In this case, when bidder 1 with $x_1 \in (\underline{x}_1, \bar{x}_1)$ loses, there is a positive probability for bidder 2 to hold a resale in the event that bidder 1 obtains second signal 1 while bidder 2 obtains second signal 0.

   (b) There exists an $\check{x} \in (\underline{x}_1, \bar{x}_1)$ such that $\beta_1(\check{x}) \neq \beta_2(\check{x})$. Without loss of generality, we assume that $\beta_1(\check{x}) < \beta_2(\check{x})$. By continuity, there exists a sufficiently small $\xi > 0$, such that $\beta_1(x) < \beta_2(x)$ for $x \in (\check{x} - \xi, \check{x} + \xi)$. When bidder 1 with $x_1 \in (\check{x} - \xi, \check{x} + \xi)$ loses, there is a positive probability that she loses to bidder 2 with $x_2 < x_1$; thus she expects resale to occur with positive probability.

   (c) $\underline{x}_1 \neq \underline{x}_2$, or $\bar{x}_1 \neq \bar{x}_2$. Without loss of generality we consider the case $\bar{x}_2 < \bar{x}_1$ only. Clearly, in this case when bidder 1 with $x_1 \in (\bar{x}_2, \bar{x}_1)$ loses, she expects resale to occur with positive probability.

■

One can easily see that the above two lemmas trivially hold in the symmetric case where $\beta_1(\cdot) = \beta_2(\cdot)$.

Proposition 1 *There does not exist any equilibrium in which at least one bidder reveals her first-auction value with positive probability.*

Proof. See Appendix. ■

In particular, Proposition 1 implies that there does not exist an equilibrium in which both bidders bid according to a symmetric and strictly increasing function in the first auction.

A similar result is established by Krishna (2002), although the environment is different. In Hafalir and Krishna (2008), they mention that the loser's bid should not be announced in an asymmetric first-price sealed bid auction with resale (monopoly offer). The reason is that, to establish an increasing equilibrium for the initial auction, the resale seller should not be able to infer the loser's value. This consideration can be interpreted alternatively: in a sequential game, a player is willing to fully reveal her type only when this information cannot be used by her competitor to harm her in a later stage. This intuition is consistent with that provided in Garratt and Tröger (2006, 2008), and is also the intuition for Lemma 2: a bidder is not willing to fully reveal her type since there is a positive probability of resale.

4 Resale with Monopsony Offer

We now consider the case in which the buyer makes the offer to the seller (monopsony offer). Again we consider second-price sealed bid auctions.

Suppose there exists a symmetric increasing equilibrium bidding strategy $\beta(x)$ for the first item. Then, loser 1, who makes a bid b, infers that winner 1's first signal follows the truncated distribution $F[\beta^{-1}(b), 1]$. We will first solve for the candidate $\beta(x)$ and then verify that no one has an incentive to deviate when the opponent follows this $\beta(x)$.

On the equilibrium path, it can be easily verified that if loser 1 follows the equilibrium strategy, the only event in which resale occurs is when she gets signal 1 in the second stage while her rival (winner 1) gets signal 0. If she does not follow the equilibrium strategy in the first stage, resale may occur in other events.

We assume that the regularity condition holds for $F(\cdot)$, that is, $(c - F(x))/f(x)$ is decreasing in x for any constant $c \in [0, F(x)]$. We will work backward.

4.1 The Resale

Suppose that in the resale stage, loser 1 makes an offer r to buy the first item from winner 1. As argued before, the optimal offer should solve:

$$\begin{aligned}
&\max \frac{F(r) - F(\beta^{-1}(b))}{1 - F(\beta^{-1}(b))}(x + \theta - r) + \frac{1 - F(r)}{1 - F(\beta^{-1}(b))} \\
&\text{s.t. } r \in [\beta^{-1}(b), 1] \\
&x + \theta - r \geq 1 \ .
\end{aligned}$$

Let $\pi(x, \beta^{-1}(b), \theta)$ denote the value of the objective function when r is offered optimally. That is, $\pi(x, \beta^{-1}(b), \theta)$ is the expected payoff to loser 1, who possesses first signal x and bids b in the first stage, conditional on resale.

It is easily seen that when r is unconstrained, its optimal value should satisfy

$$x + \theta - 1 = r + \frac{F(r) - F(\beta^{-1}(b))}{f(r)}. \quad (1)$$

Given the regularity condition, the right hand side of the above equation is increasing in r. Hence the solution is unique. It is easily verified that as long as $r \geq \beta^{-1}(b)$, then $x + \theta - r \geq 1$. So the buyer can only be better off with resale.

Let $r(x, \beta^{-1}(b))$ be the solution to the unconstrained FOC. It can be verified that $r(x, \beta^{-1}(b))$ is increasing in both x and $\beta^{-1}(b)$. Let $r^*(x, \beta^{-1}(b))$ denote the solution with constraints. Then

$$r^*(x, \beta^{-1}(b)) = \min\{1, \max\{r(x, \beta^{-1}(b)), \beta^{-1}(b)\}\}.$$

Given x, the LHS of equation (1) is increasing in θ. This implies that there exists a $\hat{\theta}$ such that when $\theta \geq \hat{\theta}$,

$$x + \theta - 1 > r + \frac{F(r) - F(\beta^{-1}(b))}{f(r)},$$

since r cannot exceed 1. Therefore $r^*(x, \beta^{-1}(b)) = 1$ for any x and $\beta^{-1}(b)$ when $\theta \geq \hat{\theta}$. By setting $r = 1, x = 0$,and $\beta^{-1}(b) = 0$, it can be shown that

$$\hat{\theta} = 2 + \frac{1}{f(1)} \quad (2)$$

When $\theta < \hat{\theta}$ and $\beta^{-1}(b) \geq g(x, \theta)$, where $g(x, \theta)$ solves $x+\theta-1 = 1+\frac{1-F(g(x,\theta))}{f(1)}$, $r^*(x, \beta^{-1}(b)) = 1$. When $\theta < \hat{\theta}$,and $\beta^{-1}(b) < g(x, \theta)$, $r^*(x, \beta^{-1}(b)) = \max\{r(x, \beta^{-1}(b)), \beta^{-1}(b)\}$.

To save notation, let $r(x) = r^*(x, x)$ denote the optimal reserve price when loser 1 follows equilibrium $\beta(\cdot)$ in the first auction ($\beta^{-1}(b) = x$). When $\beta^{-1}(b) = x$, the condition $\beta^{-1}(b) \geq g(x, \theta)$ can be rewritten as $x \geq g(\theta)$, where $g(\theta)$ solves $g(\theta) + \theta - 1 = 1 + \frac{1-F(g(\theta))}{f(1)}$.

When we check deviations, we will consider off-equilibrium resales and assume that the potential seller comes to resale whenever the potential buyer holds the item. To consider the perfect Bayesian equilibrium (PBE), we also assume that the potential buyer's beliefs follow the Bayesian rule off the equilibrium path whenever it applies.

4.2 The Second Auction

We will consider the following strategies and beliefs in the second auction:

When the second signal $y = 1$, winner 1 bids θ, and loser 1 bids $\pi(x, \beta^{-1}(b), \theta)$.[3] When the second signal $y = 0$, both winner 1 and loser 1 bid 0.

Loser 1's belief is that when winner 1 loses the second item or wins it in a tie at price 0, winner 1's second signal is 0; when loser 1, with a second signal 1, loses against winner 1, she believes that winner 1's second signal is 1. Winner 1's belief does not matter (since she does not make offers in the resale stage). Loser 1 holds a resale when her expected net gain from resale is positive and the potential seller accepts the offer whenever she is weakly better off.

Note that when $\beta^{-1}(b) = x$ or close to x, $\pi(x, \beta^{-1}(b), \theta) < \theta$. So when winner 1 and loser 1 both have second signal 1, winner 1 will obtain the second item in equilibrium.

There seems to be an incentive for deviation where winner 1 with $y = 0$ may overbid loser 1 when the latter has $y = 1$ in order to resell the whole package back to loser 1. However, assuming that loser 1 follows the equilibrium strategy, winner 1 does not expect loser 1 to hold a resale in this case so there is no gain for winner 1 when overbidding.

4.3 The First Auction

We consider a bidder with a first signal x who makes a bid b for the first item. Suppose her rival has a first signal z and follows the equilibrium strategy. For illustration purpose we assume that $x \leq r(z)$, which implies that $r^{-1}(x) \leq z$. When $z \in [r^{-1}(x), \beta^{-1}(b)]$, the offer $r(z)$ will be accepted by the resale seller (the bidder in question) when she loses the second item. Since $x \leq r(x)$ implies $r^{-1}(x) \leq x$, at the equilibrium where $x = \beta^{-1}(b)$, we cannot have $\beta^{-1}(b) < r^{-1}(x)$. Also note that winner 1 can infer loser 1's first signal; thus she knows whether or not she will accept the buyer's offer in resale (if any) immediately after the first auction.

Proposition 2 *The strategies and beliefs specified in the previous two subsections for the resale and the second auction, and the following symmetric bid function, constitute a Perfect Bayesian Equilibrium:*

$$\beta(x) = p^2(x + \theta) + (1 - p)^2 x + 2p(1 - p)r(x) - p^2\pi(x, x, \theta), \quad (3)$$

[3] Note that winner 1's payment equals loser 1's bid in the first auction, and thus she can infer loser 1's first signal.

where

$$r(x)=\begin{cases} 1, \textit{ when } \theta\geq\hat{\theta}, \textit{ or } \theta<\hat{\theta} \textit{ and } x\geq g(\theta) \\ \textit{the unique solution of } x+\theta-1=r+\frac{F(r)-F(x)}{f(r)}, \textit{ otherwise} \end{cases}, \textit{ and}$$

$$\pi(x,x,\theta)=\begin{cases} x+\theta-1, \textit{ when } \theta\geq\hat{\theta}, \textit{ or } \theta<\hat{\theta} \textit{ and } x\geq g(\theta) \\ \frac{F(r(x))-F(x)}{1-F(x)}(x+\theta-r(x))+\frac{1-F(r(x))}{1-F(x)}, \textit{ otherwise} \end{cases}.$$

Proof. See Appendix. ■

The proof is tedious, so it is relegated to the appendix. To summarize, given the strategies and beliefs in the resale and the second auction specified in the previous subsections, we first identify a candidate symmetric equilibrium bid function $\beta(\cdot)$ for the first item, and then proceed with the following steps: first, we show that if both bidders followed the proposed bidding strategy in the first auction, no bidder has incentives to deviate in the second auction and the resale stage; we then show that no bidder has an incentive to deviate from $\beta(\cdot)$ in the first auction and plays optimally, conditional on this deviation, in the following stages. The first step is straightforward to show, and our proof in the appendix focuses on the second step.

Note that there are multiple equilibria in this new setting with monopsony offers. For example, based on the PBE proposed in Proposition 2, one can obtain infinitely many other PBE's by perturbing winner 1's strategy, so that when her second signal is 0, she bids any positive amount less than $\pi(x^{-i},x^{-i},\theta)$ rather than 0 in the second auction, where x^{-i} is her competitor's first signal inferred from the first auction payment. It follows immediately that she does not have an incentive to deviate from this strategy, given that the other bidder follows the strategy specified in Proposition 2. Similarly, when loser 1's second signal is 0, she can bid any amount between 0 and θ as long as winner 1 bids 0 for the second item when her second signal is 0. Note, however, this multiple equilibria problem is caused by the discreteness of the second auction signals.

In Proposition 2, we focus on the equilibrium in which both bidders bid 0 when their second signal is 0.[4] In the proof, when we consider the events off equilibrium path, we utilize the fact that as long as loser 1 follows the proposed strategy and wins the second item when her second signal is 1, she believes that winner 1's second signal is 0 regardless of her payment for the second item.

[4] This corresponds to the "bid your value" strategy in the one-shot counterpart of the second-price sealed bid auction.

4.4 Resale vs. Non-resale: A Comparison

First, although loser 1 bids more aggressively when her second signal is 1 which results in a higher revenue for the seller in the second auction, the resale effect imposed on the bidding function for the first auction is ambiguous. In the appendix, we show that there is no general ranking between the bidding functions for the first auction with and without resale.

This suggests that the revenue comparison is also ambiguous, although the allocation efficiency is apparently higher when resale is allowed. To see this more specifically, let R^{NR} and R^{R} denote the seller's expected revenue without and with resale, respectively. Then the seller's expected revenue is the sum of the expected revenues from the two auctions. It can be shown that:

$$\begin{aligned} R^{NR} &= \int_0^1 [x + p(\theta - 1) + p^2] \cdot 2f(x)(1 - F(x))dx, \\ R^{R} &= \int_0^1 [p^2(x + \theta) + (1 - p)^2 x + 2p(1 - p)r(x)] \cdot 2f(x)(1 - F(x))dx. \end{aligned}$$

For comparison purpose, we define the difference in expected revenue between the two cases as follows:

$$\begin{aligned} DR &= R^{NR} - R^{R} \\ &= \begin{cases} p(1-p)\{\int_0^1 (\theta + 2x - 3) \cdot 2f(x)(1 - F(x))dx + \int_0^{g(\theta)} [2 - 2r(x)] \cdot 2f(x)(1 - F(x))dx\}, \text{when } \theta < \hat{\theta} \\ p(1-p)\int_0^1 (\theta + 2x - 3) \cdot 2f(x)(1 - F(x))dx, \text{ otherwise} \end{cases} \end{aligned}$$

Clearly, when $\theta \geq \hat{\theta} = 2 + \frac{1}{f(1)}$, $R^{NR} \geq R^{R}$ if $Ex^{(2)} \geq \frac{3-\theta}{2}$, and $R^{NR} < R^{R}$ if $Ex^{(2)} < \frac{3-\theta}{2}$, where $x^{(2)}$ is the second order statistic. Therefore, given the distribution $F(\cdot)$ (so $Ex^{(2)}$ is fixed), when θ is sufficiently large, the seller will be better off without resale.

When $\theta < \hat{\theta}$, the analysis becomes complicated. Since $g(\theta)$ and $r(x)$ are endogenously determined, it is difficult to determine the sign of DR based on θ, p, and the distribution. However, it is obvious that when $\theta = 1, DR = 0$. Thus, one can check the sign of $\frac{\partial DR}{\partial \theta}$ as $\theta \to 1^+$. Clearly, if $\frac{\partial DR}{\partial \theta}|_{\theta \to 1^+} < 0$ for some distribution, $DR < 0$ as $\theta \to 1^+$, and the seller's expected revenue would be lower without resale.

It can be shown that $\frac{\partial DR}{\partial \theta}|_{\theta \to 1^+} = p(1-p)\left[1 - 2\int_0^1 \frac{2f(x)(1-F(x))}{VR'(x)}dx\right]$, where $VR(x) = x - \frac{1-F(x)}{f(x)}$. Hence $\frac{\partial DR}{\partial \theta}|_{\theta \to 1^+} < 0$ if $\int_0^1 \frac{2f(x)(1-F(x))}{VR'(x)}dx > \frac{1}{2}$, which holds when $E\frac{1}{VR'(x^{(2)})} > \frac{1}{2}$. This means that when $\theta < 2 + \frac{1}{f(1)}$ and $E\frac{1}{VR'(x^{(2)})} > \frac{1}{2}$, the seller's expected revenue is higher when resale is allowed.

In what follows, we focus on the generalized uniform distribution family, namely, $F(x) = x^{\alpha}, \alpha > 0$.

When $\theta \geq \hat{\theta} = 2 + \frac{1}{\alpha}, DR = p(1-p)\frac{2\alpha^3-\alpha^2+2\alpha+1}{\alpha(\alpha+1)(2\alpha+1)} > 0$. Therefore, for any generalized uniform distribution, the seller's expected revenue is higher without resale when $\theta \geq 2 + \frac{1}{\alpha}$.

When $\theta < \hat{\theta}, \frac{\partial DR}{\partial \theta} = p(1-p)\left\{1 + \int_0^{g(\theta)} \frac{-2f^2(r(x))}{2f^2(r(x))+f'(r(x))[1-F(r(x))]} \cdot 2f(x)(1-F(x))dx\right\}$. When $f^{'}(\cdot) \geq 0$, one can see that $\frac{-2f^2(r(x))}{2f^2(r(x))+f'(r(x))[1-F(r(x))]} \geq -1$, and thus

$$\begin{aligned}
& \int_0^{g(\theta)} \frac{-2f^2(r(x))}{2f^2(r(x)) + f'(r(x))[1-F(r(x))]} \cdot 2f(x)(1-F(x))dx \\
> & \int_0^{1} \frac{-2f^2(r(x))}{2f^2(r(x)) + f'(r(x))[1-F(r(x))]} \cdot 2f(x)(1-F(x))dx \\
\geq & -1.
\end{aligned}$$

Therefore, when $f^{'}(\cdot) \geq 0$, namely when $\alpha \geq 1$, then $\frac{\partial DR}{\partial \theta} > 0$ for $\theta < 2 + \frac{1}{\alpha}$.

When $f^{'}(\cdot) < 0$ and $\theta < 2 + \frac{1}{\alpha}$, it is hard to characterize a general condition, so we check the case when $\theta \to 1^+$.

$$\frac{\partial DR}{\partial \theta}|_{\theta \to 1^+} = \frac{2\alpha}{(\alpha+1)^3}\left\{\frac{3}{2}\alpha^2 + \alpha - \frac{1}{2} - 2\alpha(\alpha-1)[\ln(2\alpha) - \ln(1-\alpha)]\right\}.$$

It can be verified that when $\alpha \in (\alpha^*, 1)$,where $\alpha^* \approx 0.7580$, $\frac{\partial DR}{\partial \theta}|_{\theta \to 1^+} < 0$, i.e., resale increases the seller's expected revenue when θ is sufficiently close to 1.

Next, we turn to the comparison of probabilities of the "exposure" problem. When resale is not allowed, the exposure problem arises when winner 1's second signal is 0, while her payment for the first item is higher than her stand-alone value for the first item; When resale is allowed, the exposure problem arises when winner 1 ends up overpaying for the first item when both bidders' second signals are 0, or when her expected payoff from resale is less than her payment for the first item.

We define $P_{Ep}^{NR}(x)$ and $P_{Ep}^{R}(x)$ to be the probabilities of exposure for the case without and with resale, respectively, given the bidder's first signal x. Let z be the other bidder's first signal. These two probabilities can be expressed as

$$
\begin{aligned}
P_{Ep}^{NR}(x) &= (1-p)\cdot P(z<x,\beta^{NR}(z)>x),\\
P_{Ep}^{R}(x) &= (1-p)^2\cdot P(z<x,\beta^{R}(z)>x)+p(1-p)\cdot P(z<x,\beta^{R}(z)>\max\{x,r(z)\}).
\end{aligned}
$$

While it is complicated to make a general comparison, we can relate the comparison of exposure probabilities to the bidding functions in the first stage. We state the following proposition:

LEMMA 3 *When $x>0$, a sufficient condition for $P_{Ep}^{NR}(x)>P_{Ep}^{R}(x)$ is $\beta^{NR}(z)\geq\beta^{R}(z)$ for $z\in[0,x)$.*

Proof. From the definition of $P_{Ep}^{R}(x)$, we know that since $P(z<x,\beta^{R}(z)>\max\{x,r(z)\})<P(z<x,\beta^{R}(z)>x)$, we have $P_{Ep}^{R}(x)<(1-p)\cdot P(z<x,\beta^{R}(z)>x)$. If $\beta^{NR}(z)\geq\beta^{R}(z)$ for $z\in[0,x)$, $P(z<x,\beta^{R}(z)>x)<P(z<x,\beta^{NR}(z)>x)$, hence $P_{Ep}^{R}(x)<P_{Ep}^{NR}(x)$. ■

As shown in the appendix (in comparing $\beta^{NR}(\cdot)$ and $\beta^{R}(\cdot)$), a sufficient condition for $\beta^{NR}(\cdot)>\beta^{R}(\cdot)$ is $\theta+p\geq 3$. Thus, when the synergy effect is sufficiently strong, $P_{Ep}^{NR}(x)>P_{Ep}^{R}(x)$ for all $x\in[0,1]$, which also implies that ex ante, the expected probability of exposure is higher when resale is not allowed.

It turns out that the condition $\theta+p\geq 3$ is "tight" in the sense that when $\theta+p<3$, we can identify conditions under which $P_{Ep}^{NR}(x)\leq P_{Ep}^{R}(x)$. To see this, we consider $\theta>\hat{\theta}=2+\frac{1}{f(1)}$ and $\theta+p<3$, in which case we have

$$
\begin{aligned}
P_{Ep}^{NR}(2p-p^2) &= (1-p)[F(2p-p^2)-F(2p-p^2-p(\theta-1))],\\
P_{Ep}^{R}(x) &= (1-p)^2F(2p-p^2).
\end{aligned}
$$

Clearly, when p is very close to 0, there exist distributions and θ such that $\theta>\hat{\theta}=2+\frac{1}{f(1)}$ and $\theta+p<3$. Meanwhile, it is trivial to show that if $p=0$, there is no exposure problem because there is no chance of getting a signal of 1 for the second item, which means $P_{Ep}^{NR}(0)=0=P_{Ep}^{R}(0)$.[5] We can also demonstrate that

$$
\lim_{p\to 0^+}P_{Ep}^{NR}(p)=(\theta-1)f(0)<2f(0)=\lim_{p\to 0^+}P_{Ep}^{R}(p),
$$

[5] Although we have not included the trivial cases of $p=0,1$, all the analysis and results are applied by continuity.

since $\theta + p < 3$ implies that $\theta < 3$.

Thus $P_{Ep}^{NR}(p) < P_{Ep}^{R}(p)$ when p is sufficiently close to 0, and that $\theta > \hat{\theta}$ and $\theta + p < 3$.

The above comparison is based on the general distribution $F(\cdot)$. Next we will focus on the case in which the first item's value is distributed uniformly over $[0, 1]$.

The Uniform Distribution Case

When the first signal follows $U[0, 1]$, it can be verified that $\hat{\theta} = 3, g(\theta) = \frac{3-\theta}{2}$, and $r(x)$ and $\pi(x, x, \theta)$ are given below.

$$r(x) = \begin{cases} 1, \text{when } \theta \geq 3 \text{ or } \theta < 3 \ \& \ x \geq \frac{3-\theta}{2} \\ x + \frac{\theta-1}{2}, \text{ otherwise} \end{cases}$$

$$\pi(x, x, \theta) = \begin{cases} x + \theta - 1, \text{when } \theta \geq 3 \text{ or } \theta < 3 \ \& \ x \geq \frac{3-\theta}{2} \\ 1 + \frac{(\theta-1)^2}{4(1-x)}, \text{ otherwise} \end{cases}$$

Therefore, the equilibrium bid function for the first item is:

$$\beta^R(x) = \begin{cases} (1-p)^2 x - p^2 + 2p, \text{when } \theta \geq 3 \text{ or } \theta < 3 \ \& \ x \geq \frac{3-\theta}{2} \\ x + p(\theta - 1) - p^2 \frac{(\theta-1)^2}{4(1-x)}, \text{ otherwise} \end{cases}$$

Since $\beta^{NR}(x) = x + p(\theta - 1)$, it can be easily verified that $\beta^R(x) < \beta^{NR}(x)$ for any $p \in (0, 1)$ and $\theta > 1$, which means that with the opportunity of resale, bidders bid less aggressively for the first item. From Lemma 3, it follows that the exposure probability is lower when resale is allowed.

Define $B^{NR}(x)$ and $B^R(x)$ to be the expected payoff for a bidder with the first signal x under the two scenarios. With some derivation, it can be shown that:

$$B^{NR}(x) = \int_0^x (x - z)dz + p(1 - p)$$

$$B^R(x) = \begin{cases} (1 - 2p + 2p^2) \int_0^x (x - z)dz + p(1 - p)(x + \theta - 1), , \theta \geq \hat{\theta} \\ (1 - p + p^2)\frac{x^2}{2} + p(1 - p)(3 - \theta)\frac{x}{2} + p(1 - p)\frac{\theta^2+1}{4}, \theta < \hat{\theta} \text{ and } x \geq g(\theta) \\ \frac{x^2}{2} + p(1 - p)(\frac{3}{8}\theta^2 - \frac{3}{4}\theta + \frac{11}{8}), \text{ otherwise} \end{cases}$$

It can be verified $B^{NR}(x) < B^R(x)$ holds for $x \in [0, 1]$.

We have argued that the effect of allowing resale on the auctioneer's revenue is ambiguous. For the uniform distribution, bidders always bid less aggressively for the first item. However, bidders bid more aggressively for the second item. With some derivations, we can show that

$$R^{NR} = p\theta + p^2 - p + \frac{1}{3}.$$

$$R^R = \begin{cases} p^2\theta + \frac{-4p^2+4p+1}{3}, \text{when } \theta \geq 3 \\ \frac{-p(1-p)\theta^3}{12} + \frac{p(1-p)\theta^2}{4} + \frac{(p^2+3p)\theta}{4} + \frac{11p^2-11p+4}{12}, \text{ when } \theta < 3 \end{cases}$$

$$\Rightarrow R^{NR} - R^R = \begin{cases} p(1-p)(\theta - \frac{5}{3}), \text{when } \theta \geq 3 \\ \frac{p(1-p)}{12}(\theta - 1)^3, \text{ when } 1 < \theta < 3 \end{cases}$$

Clearly, $R^{NR} > R^R$, which implies that in this uniform distribution case, the effect of underbidding for the first item dominates the effect of overbidding for the second item.

5 Concluding Remarks

This paper offers the first analysis of auctions with synergies by explicitly considering the post-auction resale opportunity, which is arguably more natural if we believe that secondary markets cannot always be banned.

Our analysis suggests that when the initial auctions take the form of second-price sealed bids, there is a potential problem of information revelation, even if bids are not announced. In particular, a monopoly mechanism in resale can destroy any equilibrium in which a bidder has to fully reveal her type over some range with positive probability. Since typically an auctioneer cannot control the selling mechanism in the resale stage, our results have some immediate policy implication on how secondary markets should be regulated.

In our model, the synergy enters the value of a package as a multiplier to the value of the second item, while the previous literature in general assumes that synergies enter the value of a package as a term to be added to the stand-alone value of a package. However, we do not think that our major results would be altered if we adopt the "summation" version instead: Inefficiency would still persist when resale was banned, and in the resale stage, the seller or buyer would still make an optimal offer, taking into account the synergy.

We assume a Bernoulli distribution for the second-auction signal in our model and consider two bidders only in order to avoid unnecessary complexity in the analysis. If the second-auction signal follows a general distribution, resale events on the equilibrium path will become tedious. If we allow for any finite number of bidders, the party making the offer in resale needs to guess the range of the first-auction signal possessed by the potential buyers (or sellers).[6] It is thus clear that the relaxation of either assumption would render our equilibrium analysis intractable; Nevertheless one direction of our future research is to generalize our current analysis. Another direction is to extend our analysis to the simultaneous auction setting, where the "exposure" problem with synergy can be more striking than in the sequential auction setting.

[6] More specifically, if there are more than two bidders, the information structure before resale is very complicated. In a monopoly offer, the winner of the first item needs to estimate the probability of the event that the bidder with the second highest first signal wins the second item. In such an event, she knows the signal precisely in an increasing equilibrium; in all the other events, her posterior about the buyer's first signal will be a truncated distribution from $F(\cdot)$ (truncated by the second highest signal). Similarly in a monopsony resale, the winner of the second item has a posterior about the ranking of her first signal among all the bidders based on the payment she made for the second item, which makes the calculation of the best offer in resale very tedious.

Appendix

Proof of Proposition 1: Without loss of generality we will consider the following strategies and beliefs for the second auction and the resale stages: winner 1 bids θ when her second signal is 1 and bids δ, a sufficiently small positive amount, when her second signal is 0; loser 1 bids 1 when her second signal is 1 and 0 otherwise. Winner 1's beliefs about loser 1's second signal are that it is 0 if she wins the second item at zero price, and 1 in any other case. In the resale, winner 1's offer is a vector of the item(s) offered for resale and the corresponding optimal price(s).[7] Winner 1 does not offer the second item for resale when she believes that loser 1's second signal is 0. Loser 1's beliefs about winner 1's signals do not matter and she accepts any offer in resale as long as she is weakly better off by doing so. It can be verified that the above pair of the strategy profile and belief system constitutes a perfect Bayesian equilibrium (PBE) for the second auction and resale stages, and in such a PBE, loser 1 plays a pure and separating strategy so that after the second auction, winner 1 can infer loser 1's second-item signal.[8]

Without loss of generality, we consider the case in which conditional on losing the first auction, bidder 1 expects resale to occur with positive probability only when the second signal is 1 for her and 0 for bidder 2. We follow this case for ease of illustration; the logic of our analysis below should go through for all the other cases discussed in the proof of Lemma 2.

Suppose bidder 1, with the first signal x, plays as if her type is $\hat{x}$ in the first auction, but plays optimally, conditional on x and $\hat{x}$, in the following two stages (i.e., the second auction and the resale). For resale, this means that if bidder 1 underbids in a strictly increasing segment and loses, she will accept the offer in resale (if any) because the seller underestimates her type; if she overbids in a strictly increasing segment and loses, she will reject the offer in resale because the resale seller

[7] Because of the asymetry in the first auction bid functions, it is possible that even if both bidders' second signals are the same, there is a potential gain from resale. Taking this into account, winner 1 calculates the optimal offer vector given her beliefs about loser 1's two signals.

[8] There are other equlibria in which winner 1 can infer loser 1's second signal after the second auction. Using backward induction, one can verify that the contingent payoffs in resale for winner 1 and loser 1 when their second signals are 1 are at least θ and 1, respectively. Moreover, we do not require winner 1 to infer loser 1's second signal in cases where conditional on winning the first auction and her beliefs about loser 1's first signal, winner 1 does not find it profitable to hold a resale for all possible subsequent events (in these cases, winner 1 can bid 0 when her second signal is 0).Working with those equilibria will not alter our analysis.

overestimates her type. Clearly, for $\beta_1(\cdot)$ to be an equilibrium bidding strategy, we require that $\hat{x} = x$ for $x \in (x_{1*}, x^{1*})$, where the interval (x_{1*}, x^{1*}) is identified in Lemma 2. Below we will only consider local deviations so that $\hat{x}$ is sufficiently close to x. By continuity, Lemma 2 implies that with a sufficiently small deviation, conditional on losing the first auction, bidder 1 still expects bidder 2 to hold a resale with positive probability.

Bidder 1 needs to solve the following problem:

$$\max_{\hat{x} \in (x_{1*}, x^{1*})} \int_0^{\beta_2^{-1}(\beta_1(\hat{x}))} [w(x,y) - \beta_2(y)] f(y) dy + \int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 l(x,y,\hat{x}) f(y) dy$$

where $w(x,y)$ is bidder 1's contingent payoff if she obtains the first item given her first signal x and her rival's first signal y, and $l(x,y,\hat{x})$ is bidder 1's contingent payoff if she loses the first item in the first auction given that she mimics type $\hat{x}$.

By backward induction, $l(x,y,\hat{x}) = p(1-p)[(x-\hat{x}) I_{\{x>\hat{x}\}} I_{\{y \le \hat{x}+\theta-1\}} + 1 - \delta]$, where $y \le \hat{x}+\theta-1$ represents the events where bidder 2 finds it profitable to hold a resale when her second signal is 0 and bidder 1's second signal is 1 and $\hat{x}+\theta-1$ is the optimal offer given her beliefs.

Now we consider the case in which bidder 1 mimics type $\hat{x}$, which is sufficiently close to x. When this deviation is downward, which means that $x > \hat{x}$, we have

$$\int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 l(x,y,\hat{x}) f(y) dy = \int_{\beta_2^{-1}(\beta_1(\hat{x}))}^{\min\{\hat{x}+\theta-1,1\}} p(1-p)[x-\hat{x}+1-\delta] f(y) dy + \int_{\min\{\hat{x}+\theta-1,1\}}^1 p(1-p) f(y)(1-\delta) dy.$$

If this deviation is upward, which means that $x < \hat{x}$, we have

$$\int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 l(x,y,\hat{x}) f(y) dy = \int_{\beta_2^{-1}(\beta_1(\hat{x}))}^1 p(1-p) f(y)(1-\delta) dy.$$

Next, we define FOC_{down} (FOC_{up}) to be the derivative w.r.t $\hat{x}$ in the downward (upward) deviation.

Letting $\hat{x} \uparrow x$ in the downward case and $\hat{x} \downarrow x$ in the upward case, we can verify that

$$\lim_{\hat{x} \uparrow x} FOC_{down} = \lim_{\hat{x} \downarrow x} FOC_{up} - p(1-p) \int_{\beta_2^{-1}(\beta_1(x))}^{\min\{\hat{x}+\theta-1,1\}} f(y) dy.$$

Thus we conclude that $FOC_{down}(b \uparrow \beta_1(x)) < FOC_{up}(b \downarrow \beta_1(x))$ for $p \in (0,1)$. This inequality suggests that when we evaluate the objective function (the expected payoff for a bidder with type x who makes a small deviation in the first stage) at $b = \beta_1(x)$, the left derivative is smaller than the right derivative. Clearly, for $\beta_1(\cdot)$ to be an equilibrium, we must have incentive compatibility condition $\hat{x} = x$. Hence $\hat{x} = x$ is at least a local maximum within a small interval around x.

It can be easily verified that the objective function is continuous at $\hat{x} = x$. For the left derivative to be smaller than the right derivative for a continuous function, it is either $FOC_{down} < FOC_{up} \leq 0$, $0 \leq FOC_{down} < FOC_{up}$, or $FOC_{down} < 0 < FOC_{up}$. For all these three cases, it can be easily verified that $\hat{x} = x$ cannot be a local maximum. ■

Proof of Proposition 2: Suppose that bidding for the second item follows the proposed equilibrium strategies. We will first compute the candidate equilibrium bid function for the first item ($\beta(\cdot)$) and then verify that it is indeed the equilibrium bid function for the first item.

Case 1: $\theta \geq \hat{\theta}$, or $\theta < \hat{\theta}$ and $\beta^{-1}(b) = x \geq g(\theta)$, which implies that the offer made by this bidder in resale should be 1.

When she wins and $\theta \geq \hat{\theta}$, the rival will make an offer $r(z) = 1$ in resale if resale occurs, where z is the rival's first auction signal;

When she wins and $\theta < \hat{\theta}$, we have

$$\begin{aligned} &\int_0^{\beta^{-1}(b)} E(payoff|winning\ the\ first\ item)f(z)dz \\ = &\int_0^{g(\theta)} E(payoff|winning\ the\ first\ item)f(z)dz + \int_{g(\theta)}^{\beta^{-1}(b)} E(payoff|winning\ the\ first\ item)f(z)dz. \end{aligned}$$

However, to derive FOCs we only need to consider the event $z \geq g(\theta)$, since the first term above does not contain b. When $z \geq g(\theta)$, she accepts the offer for sure since $r(z) = 1$.

The expected gross payoff upon winning is given by

$$p^2[x + \theta - (z + \theta - 1)] + (1 - p)p(x + \theta) + p(1 - p) + (1 - p)^2 x$$

and the expected payoff upon losing is given by

$$p(1 - p)(x + \theta - 1).$$

It can be easily verified that the implication of the FOCs in this case is that a bidder bids an amount equal to her maximal willingness to pay, which is the difference between the expected payoffs conditional on winning and not winning the item when a tie occurs.

Straightforward calculations lead to

$$\beta(x) = (1-p)^2 x - p^2 + 2p. \tag{4}$$

Clearly, in this case, $\beta'(x) > 0$.

Case 2: $\theta < \hat{\theta}$ and $\beta^{-1}(b) = x < g(\theta)$.

In this case, loser 1 will make an offer less than 1 in resale. Again, when a bidder with $x < g(\theta)$ wins, the only event that is relevant for FOCs is when her rival's signal z satisfies $r(z) \geq x$.

The gross contingent payoff upon winning for the relevant event is given by

$$p^2[x + \theta - \pi(z, z, \theta)] + (1-p)p(x+\theta) + p(1-p)r(z) + (1-p)^2 x.$$

The expected payoff upon losing is given by

$$p(1-p)\pi(x, \beta^{-1}(b), \theta).$$

Taking the derivative w.r.t b and evaluating this derivative at $\beta^{-1}(b) = x$, we have

$$\beta(x) = p^2(x+\theta) + (1-p)^2 x + 2p(1-p)r(x) - p^2\pi(x, x, \theta) \tag{5}$$

and

$$\begin{aligned} \beta'(x) &= p - \; p(1-p) + (1-p)^2 + 2p(1-p)r'(x) \\ &\quad -p^2 \frac{F(r(x)) - F(x)}{1-F(x)} \left[-\frac{f(x)}{f(r(x))} \frac{1-F(r(x))}{1-F(x)} + 1 \right]. \end{aligned}$$

Note that $\frac{1-F(v)}{f(v)}$ decreases in v and $x \leq r(x)$ implies that

$$\frac{f(x)}{f(r(x))} \frac{1-F(r(x))}{1-F(x)} \in [0, 1],$$

which further implies

$$\frac{F(r(x)) - F(x)}{1-F(x)} \left[-\frac{f(x)}{f(r(x))} \frac{1-F(r(x))}{1-F(x)} + 1 \right] \in [0, 1].$$

Therefore,

$$\begin{aligned}\beta'(x) &\geq p - \ p(1-p) + (1-p)^2 + 2p(1-p)r'(x) - p^2 \\ &= (1-p)^2 + 2p(1-p)r'(x) \\ &> 0.\end{aligned}$$

We can thus summarize (4) and (5) into (3), which is the candidate equilibrium bid function in the first auction.

Next, we prove that no bidder can deviate from the proposed equilibrium profitably. We will proceed in two steps. The first step is to show that if both bidders follow the proposed bidding strategy in the first auction, no bidder has an incentive to deviate in the second auction or in the resale stage. The second step is to show that no bidder has an incentive to deviate in the first auction and plays optimally subsequently. The first step is straightforward to verify, hence we will focus on the second step below.

Denote $V(x, \hat{x})$ to be the expected payoff for a bidder of type (first signal) x who mimics type $\hat{x}$ in the first auction and plays optimally in the second auction and the resale stage.

We first consider off-equilibrium events. Again, we assume that the potential seller shows up in the resale and accepts the offer when she is weakly better off by doing so. The potential buyer makes an offer in resale when her expected net payoff is positive. Off-equilibrium resales might involve selling both items.

If the bidder overbids and wins while $x < z$, where z is her rival's first signal (she knows z conditional on winning), she may want to pretend that her second signal is 0 when it is actually 1 in order to resell the first item to her rival when her rival also has a second signal of 1. She can do this by bidding ε, which is close to 0, when her second signal is 1; thus she wins against a rival with signal 0, but loses against a rival with signal 1 (her rival still believes her second signal is 0 even if the rival pays ε for the second item). Note that when both bidders have a signal 1, if she bids θ and wins, her rival does not hold a resale and her payoff is $x + \theta - \pi(z, z, \theta)$; if she loses the second item, she accepts $r(z)$ in resale because $z > x > r^{-1}(x)$. Therefore, when $x + \theta - \pi(z, z, \theta) < r(z)$, she will bid ε when her second signal is 1. Also note that $x + \theta - \pi(z, z, \theta) > r(z)$ when x is just a little below z, since $z + \theta - \pi(z, z, \theta) > r(z)$. When both bidders get second signal 0, although the overbidding bidder wants to sell the first item to her rival, the rival does not hold a resale.

Therefore, when $x < \hat{x}$, the gross winning payoff is given by

$$p^2 \max\{x + \theta - \pi(z, z, \theta), r(z)\} + p(1-p)(x + \theta) + p(1-p)\max\{r(z), x\} + (1-p)^2 x.$$

The expression for the losing payoff remains unchanged.

On the other hand, if the bidder underbids and loses, she holds a resale even when they both have the same second signal, since it might be the case that $x > z$, and hence her expected net payoff from resale is positive. Note that when they both have second signal 1, the bidder who underbids in the first auction would prefer to buy the whole package in the resale rather than winning the second item. In the case with underbidding, the expression for the gross winning payoff is the same as before, which is given by

$$p^2[x + \theta - \pi(z, z, \theta)] + p(1-p)(x + \theta) + p(1-p)\max\{r(z), x\} + (1-p)^2 x.$$

However, the losing payoff is now

$$p^2 l^1(x, \hat{x}, \theta) + p(1-p)\pi(x, \hat{x}, \theta) + (1-p)^2 l^0(x, \hat{x}, \theta),$$

where $l^1(x, \hat{x}, \theta)$ and $l^0(x, \hat{x}, \theta)$ denote the off-equilibrium payoffs when she holds a resale, knowing that both bidders have the same second signal 1 and 0, respectively. It is easy to verify that in these two off-equilibrium events, if $r^0(x, \hat{x})$ ($r^1(x, \hat{x})$) denotes the optimal resale offer when they both have a 0(1) second signal, $r^1(x, \hat{x}) = r^0(x, \hat{x}) + \theta$. Also, $r^0(x, \hat{x}) \in [\hat{x}, x]$.

We now show that neither overbidding nor underbidding is profitable.

Overbidding case: $x < \hat{x}$.

When $x + \theta - \pi(z, z, \theta) < r(z)$ (overbids by a relatively large amount),

$$\begin{aligned}
\frac{\partial V(x, \hat{x})}{\partial \hat{x}} &= f(\hat{x})[p^2 r(\hat{x}) + (1-p)p(x + \theta) + p(1-p)r(\hat{x}) \\
&\quad + (1-p)^2 x - \beta(\hat{x}) - p(1-p)(x + \theta - r(x, \hat{x})] \\
&= f(\hat{x})\{(1-p)^2(x - \hat{x}) + p(1-p)[r(x, \hat{x}) - r(\hat{x})] + p^2[r(\hat{x}) + \pi(\hat{x}, \hat{x}, \theta) - \hat{x} - \theta]\} \\
&< 0.
\end{aligned}$$

The inequality results from the fact that $x < \hat{x}$, $r(x, \hat{x}) \leq r(\hat{x}, \hat{x}) = r(\hat{x})$ and $\pi(\hat{x}, \hat{x}, \theta) \leq \hat{x} + \theta - r(\hat{x})$.

When $x + \theta - \pi(z, z, \theta) \geq r(z)$ (overbids by a relatively small amount),

$$\begin{aligned}\frac{\partial V(x,\hat{x})}{\partial \hat{x}} &= f(\hat{x})\{(1-p)^2(x-\hat{x}) + p(1-p)[r(x,\hat{x}) - r(\hat{x})] + p^2(x-\hat{x})\} \\ &< 0\end{aligned}$$

Therefore, overbidding is not a profitable deviation.

Underbidding case: $x > \hat{x}$.

$$\begin{aligned}\frac{\partial V(x,\hat{x})}{\partial \hat{x}} &= f(\hat{x})\{p^2[x + \theta - \pi(\hat{x},\hat{x},\theta)] + (1-p)p(x+\theta) \\ &\quad + p(1-p)\max\{r(\hat{x}), x\} + (1-p)^2 x - \beta(\hat{x}) - p^2[x + \theta - r^1(x,\hat{x})] \\ &\quad - p(1-p)[x + \theta - r(x,\hat{x})] - (1-p)^2[x - r^0(x,\hat{x})]\} \\ &= f(\hat{x})\{p(1-p)r(x,\hat{x}) - 2p(1-p)r(\hat{x}) + [p + (1-p)^2][r^0(x,\hat{x}) - \hat{x}] \\ &\quad + p(1-p)\max\{r(\hat{x}), x\}\}\end{aligned}$$

When $r(\hat{x}) \geq x$,

$$\frac{\partial V(x,\hat{x})}{\partial \hat{x}} = f(\hat{x})\{p(1-p)[r(x,\hat{x}) - r(\hat{x})] + [p + (1-p)^2][r^0(x,\hat{x}) - \hat{x}]\} > 0$$

The inequality results from the fact that $r(x, \hat{x}) \geq r(\hat{x}, \hat{x}) = r(\hat{x})$ and $r^0(x, \hat{x}) > \hat{x}$.

When $r(\hat{x}) < x$,

$$\frac{\partial V(x,\hat{x})}{\partial \hat{x}} = f(\hat{x})\{p(1-p)[r(x,\hat{x}) - 2r(\hat{x}) + x] + [p + (1-p)^2][r^0(x,\hat{x}) - \hat{x}]\} > 0 ,$$

because $r(x, \hat{x}) - 2r(\hat{x}) + x = r(x, \hat{x}) - r(\hat{x}) + x - r(\hat{x}) > 0$ and $r^0(x, \hat{x}) > \hat{x}$.

Therefore, underbidding is not a profitable deviation either. ■

The effect of resale on the bid function in the first auction

The equilibrium bid functions under the cases without resale (NR) and with resale (R) are as follows:

$$\beta^{NR}(x) = x + p(\theta - 1),$$
$$\beta^{R}(x) = p^2(x+\theta) + (1-p)^2 x + 2p(1-p)r(x) - p^2\pi(x,x,\theta).$$

Denote $\beta_1^R(x)$ for the case $\theta < \hat{\theta}$ and $x < g(\theta)$, and $\beta_2^R(x)$ for the all the other cases. We will compare $\beta^{NR}(x)$ and $\beta^R(x)$ in order.

Case 1: $\theta \geq \hat{\theta}$.

Since $\beta^{NR}(x)$ and $\beta_2^R(x)$ are linear, if they have one intersection, it must be unique; thus it suffices to check the low and high ends of these functions. At the high ends,

$$\beta^{NR}(1) = 1 + p(\theta - 1) > 1 = \beta_2^R(1).$$

When $p \leq 1 - \frac{1}{f(1)}, 3 - p \geq \hat{\theta}$. If $\theta \in (3-p, +\infty)$, then $\beta^{NR}(0) > \beta_2^R(0)$; thus $\beta^{NR}(x) > \beta_2^R(x)$ for $x \in [0,1]$.

If $\theta \in [\hat{\theta}, 3-p]$, then $\beta^{NR}(0) \leq \beta_2^R(0)$; thus $\exists | x^* \in [0,1)$ such that $\beta^{NR}(x^*) = \beta_2^R(x^*)$.

When $p > 1 - \frac{1}{f(1)}, 3 - p < \hat{\theta}$. For $\theta \in [\hat{\theta}, +\infty)$, $\beta^{NR}(x) > \beta_2^R(x)$ for $x \in [0,1]$.

Case 2: $\theta < \hat{\theta}$.

In this case,

$$\beta^R(x) = \begin{cases} \beta_2^R(x), \ x \geq g(\theta) \\ \beta_1^R(x), \ x < g(\theta) \end{cases}$$

Note that when $x < g(\theta)$, $r(x) < 1$ and $\pi(x,x,\theta) > x + \theta - 1$. The first inequality follows immediately and the second inequality results from the fact that $\pi(x,x,\theta)$ is the contingent resale payoff for loser 1 when she offers $r(x)$ in resale, and $x+\theta-1$ is the contingent resale payoff when she offers 1 instead. That she prefers to offer $r(x)$ indicates that $\pi(x,x,\theta) > x + \theta - 1$.

Therefore, $\beta_1^R(x) < \beta_2^R(x)$ for $x < g(\theta)$. For the first order comparison, we have shown earlier in the appendix that $\beta_1'^R(x) > \beta_2'^R(x)$. However, it is generally impossible to compare $\beta_1'^R(x)$ with $\beta'^{NR}(x) = 1$ based only on explicit restrictions of parameters and the distribution $F(\cdot)$.

When $p \leq 1 - \frac{1}{f(1)}, 3 - p \geq \hat{\theta}$. We have $\beta^{NR}(0) \leq \beta_2^R(0)$ and $\exists | x^* \in [0,1)$ such that $\beta^{NR}(x^*) = \beta_2^R(x^*)$.

If $x^* \geq g(\theta)$ and $\beta^{NR}(0) \in [\beta_1^R(0), \beta_2^R(0)]$, there exists at least one $\tilde{x} < g(\theta)$ such that $\beta^{NR}(\tilde{x}) = \beta_1^R(\tilde{x})$. Or equivalently, there exist at least two intersections between $\beta^{NR}$and β^R.

If $x^* \geq g(\theta)$ and $\beta^{NR}(0) \in (0, \beta_1^R(0))$, there exists at least one intersection, which is x^*.

If $x^* < g(\theta)$ and $\beta^{NR}(0) \in [\beta_1^R(0), \beta_2^R(0)]$, there is ambiguity.

If $x^* < g(\theta)$ and $\beta^{NR}(0) \in (0, \beta_1^R(0))$, there exists at least one $\tilde{x} < g(\theta)$ such that $\beta^{NR}(\tilde{x}) = \beta_1^R(\tilde{x})$.

When $p > 1 - \frac{1}{f(1)}, 3 - p < \hat{\theta}$.

If $\theta \in [3 - p, \hat{\theta})$,then $\beta^{NR}(0) \geq \beta_2^R(0)$, $\beta^{NR}(x) > \beta^R(x)$ for $x \in [0, 1]$.

If $\theta \in (1, 3 - p)$,then $\beta^{NR}(0) < \beta_2^R(0)$ and the cases are similar to those under $p \leq 1 - \frac{1}{f(1)}$.

It is clear that there is no general ranking between the bidding functions for the first auction with and without resale.

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